The Rocket Equation

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These notes are concerned with the motion of a rocket, which is propelled forwards by the expulsion of mass at the rear with a fixed velocity relative to the rocket body. Here we attempt to explain the dynamics of the rocket.

Consider a mass m travelling in a straight line with a velocity v_0. Its momentum is therefore

$$p_{tot}=m v_{0}. \tag{1}$$

Figure 1

After a time interval of δt, the mass splits into a mass $m-\delta m$ and a small mass δm, (Figure 1) and the mass δm is instantaneously projected backwards with a velocity u with respect to the mass $m-\delta m$. As a result, the momentum of the mass δm will now be

$$p_{\delta m}=\delta m(v_0-u) \tag{2}$$

and the momentum of the mass $m-\delta m$ will be

$$p_{m-\delta m}=(m-\delta m)v_0+\delta m u, \tag{3}$$

which we can write as

$$p_{m-\delta m}=(m-\delta m)(v_0+\delta m u/(m-\delta m)), \tag{4}$$

where the expression $(v_0+\delta m u/(m-\delta m))$ is evidently the velocity of the mass $m-\delta m$. This result follows from the conservation of the total system momentum, which remains constant at p_{tot}.

We now discard mass δm as 'lost' to consider the mass $m-\delta m$ in isolation and imagine, after another interval δt, another backward projection of a small mass δm, at a velocity of u relative to the remaining mass $m-2\delta m$, and again calculate the momentum changes. The momentum of mass δm is

$$p_{\delta m}=\delta m(v_0+\delta m u/(m-\delta m)-u), \tag{5}$$

and the momentum of the mass $m-2\delta m$ turns out to be

$$p_{m-2\delta m}=(m-2\delta m)(v_0+\delta m u/(m-\delta m))+\delta m u, \tag{6}$$

or

$$p_{m-2\delta m}=(m-2\delta m)(v_0+\delta m u/(m-\delta m)+\delta m u/(m-2\delta m)). \tag{7}$$

If we continue with this procedure of ejecting a small mass δm in a sequence of time steps $\delta t,$ while concentrating on the remaining mass rather than the mass ejected we can build up the following table to a limit of n steps.

Table 1

Time	Momentum of remaining mass
0	$m v_0$
δt	$(m-\delta m)(v_0+\delta m u/(m-\delta m))$
$2\delta t$	$(m-2\delta m)(v_0+\delta m u/(m-\delta m)+\delta m u/(m-2\delta m))$
$3\delta t$	$(m-3\delta m)(v_0+\delta m u/(m-\delta m)+\delta m u/(m-2\delta m)+\delta m u/(m-3\delta m))$
$etc.$	$etc.$
$n\delta t$	$(m-n\delta m)\left(v_0+\delta m u\sum_{i=1}^{n} 1/(m-i\delta m)\right)$

The final entry in this table summarises the 'rocket' action we have been investigating. We can readily identify a mass term $(m-i\delta m)$ and a velocity term $\left(v_0+\delta m u\sum_{i=1}^{n} 1/(m-i\delta m)\right)$. We shall discuss each in turn.

We may introduce the time step δt into the mass term as follows

$$m(i\delta t)=m-i\delta t\left(\frac{\delta m}{\delta t}\right). \tag{8}$$

Clearly we can write $t=i\delta t$ as the elapsed time after i steps and $\delta m/\delta t$ as the rate of fuel consumption, which is a constant we designate as $k.$ We can therefore write equation (8) as

$$m(t)=(m_0-kt), \tag{9}$$

where we have defined the time dependent mass as $m(t),$ and the initial mass as $m_0,$ which was formerly written as $m.$

We can now write the velocity term as

$$v(t)=v_0+\delta t k u\sum_{i=1}^{n} 1/m(t_i), \tag{10}$$

where t_i is the discrete time sampled at intervals of δt, and v_0 is the original velocity. In the limit of $\delta t \to 0$, we may express (10) using an integral:

$$v(t) = v_0 + k u \int_{\tau=0}^{t} (m_o - k\tau)^{-1} d\tau\,, \tag{11}$$

where τ is a time variable. Hence on integrating we obtain

$$v(t) = v_0 - u[\log(m_o - k\tau)]_0^t, \tag{12}$$

from which we find

$$v(t) = v_0 + u \log\left(\frac{m_0}{m_o - k t}\right). \tag{13}$$

This is the rocket equation, which describes how a rocket's velocity changes with time. Clearly the condition $m_0 > kt$ must hold at all times.

From equations (9) and (12) it is apparent that at time t the momentum of the rocket (plus remaining fuel) is

$$p = (m_0 - k t)\, v(t) = (m_0 - k t)\left(v_0 + u \log\left(\frac{m_0}{m_o - k t}\right)\right). \tag{14}$$

The corresponding change in momentum, Δp, (of the mass $(m_0 - kt)$,) during the time t is

$$\Delta p = (m_0 - k t)(v(t) - v_0) = (m_0 - k t) u \log\left(\frac{m_0}{m_o - k t}\right). \tag{15}$$

This is the increase in the momentum of the residual rocket mass (including fuel load at the time). The change in momentum of the exhaust gasses during powered flight, according to the principle of momentum conservation, is the negative of this.

The kinetic energy the rocket acquires in time t is

$$K_r = \frac{1}{2}(m_0 - kt)(v(t) - v_0)^2 = \frac{1}{2}(m_0 - k t) u^2 \left(\log\left(\frac{m_0}{m_o - k t}\right)\right)^2. \tag{16}$$

The kinetic energy the exhaust gas acquires in the same time is

$$K_e = \frac{1}{2kt}(\Delta p)^2 = \frac{(m_0 - kt)^2}{2kt} u^2 \left(\log\left(\frac{m_0}{m_o - kt} \right) \right)^2, \tag{17}$$

where kt is the mass of the fuel used in the time t. This is the kinetic energy of the bulk of the gas and does not account for the internal motion (i.e. the thermal energy) of the body of the gas. This can be obtained from the mean temperature of the exhaust gas, its bulk mass and composition, as in the following order of magnitude estimate:

$$K_T = \frac{3}{2} kt \left(\frac{k_B (T - T_0)}{\mu} \right), \tag{18}$$

which assumes the final products of the rocket fuel are ideal gases. The term μ is the average mass of the atoms in the gas, kt is the mass of the expended fuel, k_B is Boltzmann’s constant, T the average absolute temperature of the exhaust gas, and T_0 is the initial temperature of the fuel. Note that this does not account for any energy lost as electromagnetic radiation nor does it recognise the molecular nature of the exhaust gas, which would require us to account for changes in chemical bonding energy.

The total energy expended during the time t is therefore

$$E = K_r + K_e + K_T = \frac{(m_0 - kt)}{2kt} m_0 u^2 \left(\log\left(\frac{m_0}{m_o - kt} \right) \right)^2 + \frac{3}{2} kt \left(\frac{k_B (T - T_o)}{\mu} \right). \tag{19}$$

In thermodynamic terms the quantities K_r and K_e can be regarded as the work (W) done by the rocket, while K_T is the change in internal energy (ΔU) of the fuel mass.

An important factor in rocketry is the energy efficiency of the rocket, ϵ, which can be obtained by combining equations (16) and (19):

$$\epsilon = 100 \frac{K_r}{E_{tot}} \ \%. \tag{20}$$

This is the percentage of the total energy that is used to propel the rocket. The efficiency is unlikely to approach 100%.

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