Fitting the Orbit of a Comet

1. Outline

The purpose of this note is to provide the formulae for fitting a comet orbit. The method outlined is based on some aspects of Gauss's approach, though it does not use all of Gauss's techniques. No derivations are provided, since these can be found in standard texts such as [1 - 3]. The purpose here is to provide a recipe for the method, without getting too deeply into the mathematics.

2. Input Data

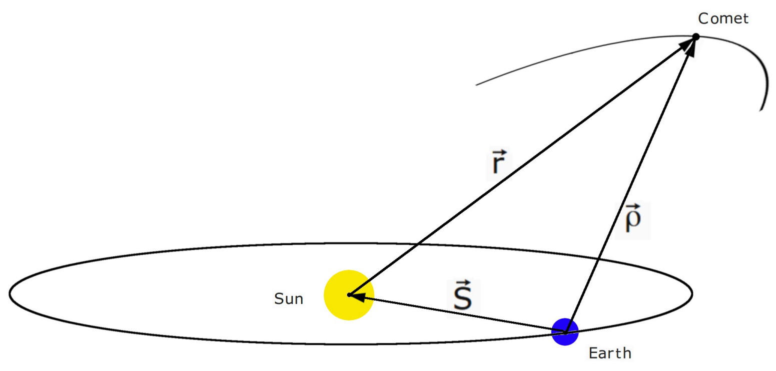

Figure 1.

The observer must supply three observations of the comet's position $\hat{O}_1, \hat{O}_2, \hat{O}_3,$ at times t_1, t_2, t_3 with $t_1 < t_2 < t_3$. The observations are defined as *unit vectors* in equatorial coordinates i.e.

$$\hat{O}_i = (\lambda_i, \mu_i, \nu_i) \quad \text{with} \quad i = 1,2,3 \tag{1}$$

where

$$\begin{aligned} \lambda_i &= \cos(\alpha_i)\cos(\delta_i), \\ \mu_i &= \sin(\alpha_i)\cos(\delta_i), \\ \nu_i &= \sin(\delta_i), \end{aligned} \tag{2}$$

and α_i is the *right ascension* (RA) and δ_i the *declination* (Dec) of the comet at time t_i. Variables λ_i, μ_i and ν_i are the so-called *direction cosines* of the observations (α_i, δ_i). Note that for these equations to be valid, both α_i and δ_i must be expressed in *decimal degrees* and not the usual hours, minutes, seconds (for RA) or degrees, minutes, seconds (for Dec). The conversions require the formulae:

$$\begin{aligned} \alpha_i &= 15(H_i + (M_i + S_i/60)/60), \\ \delta_i &= d_i + (m_i + s_i/60)/60, \end{aligned} \tag{3}$$

where H_i, M_i, S_i are the hours, minutes and seconds of RA and d_i, m_i, s_i are the degrees, minutes and seconds of Dec. For computer use, the RA and Dec should be expressed in radians, which requires multiplying the results of (3) by $\pi/180$.

In addition to the three comet positions, the observer must also supply three positions

$\vec{S}_1, \vec{S}_2, \vec{S}_3$ for the Sun corresponding to the times t_1, t_2, t_3 at which the comet was observed i.e.

$$\vec{S}_i = (X_i, Y_i, Z_i) \quad \text{with} \quad i = 1,2,3 \tag{4}$$

where

$$\begin{aligned} X_i &= R_i \cos(\mathrm{A}_i)\cos(\Delta_i), \\ Y_i &= R_i \sin(\mathrm{A}_i)\cos(\Delta_i), \\ Z_i &= R_i \sin(\Delta_i), \end{aligned} \tag{5}$$

and R_i is the Earth-Sun distance in *astronomical units* (a.u.)[1], A_i is the RA of the Sun and Δ_i is the Dec of the Sun at time t_i. To use formulae (5) the RA and Dec must again be converted to degrees or radians as appropriate. The positions of the Sun supplied here are not actual observations. They are values taken from a solar ephemeris, which exists either in tabulated form or as output from an ephemeris program. In the latter case the data are often available directly as the X_i, Y_i, Z_i coordinates in astronomical units.

Some points to note are the following:

- The times t_1, t_2, t_3 are expressed in *decimal days* measured from a specific time epoch, which is usually the first day of the calendar year or even the first day of the century. Alternatively the so-called Julian date, which commences from the Greenwich mean noon of January 1st 4713 BC, can be used. Online converters are available to change ordinary dates and times into Julian dates.
- The observed coordinates (α_i, δ_i) of the comet must, of course, be as accurate as possible and should be corrected for displacement of the observer from Earth's centre, atmospheric refraction, aberration, nutation *etc.* How these corrections can be done is described in [4].
- The RA and Dec of the comet and the Sun must be specified for the same epoch. (Beware this may not necessarily be the case if they have been obtained using different star maps and/or software).

3. The Fitting Method

The first equation required describes the positions of the comet with respect to the Sun (see Figure 1), which is formally given by the equation

$$\vec{r}_i = \rho_i \hat{O}_i - \vec{S}_i, \tag{6}$$

where ρ_i is the (initially unknown) distance from the Earth to the comet and

$$r_i = (x_i, y_i, z_i) \tag{7}$$

is the position vector locating the comet with respect to the Sun. Vectors $\vec{r}_i$ cannot be defined until the corresponding distances ρ_i have been determined. This task is central to the fitting of the orbit.

Next, Gauss defined so-called *area triangles* (see Figure 2) which specify the area between pairs of vectors taken from the three position vectors $\vec{r}_i$. These are written

1 Where 1 a.u. is 149.5979x10^6 km.

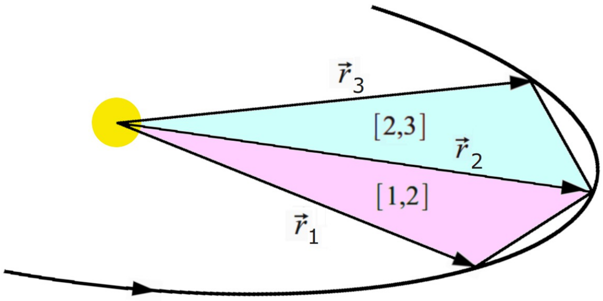

Figure 2.

as

$$[1,2]=\frac{1}{2}|\vec{r}_1\times\vec{r}_2,|$$
$$[2,3]=\frac{1}{2}|\vec{r}_2\times\vec{r}_3,| \quad (8)$$
$$[1,3]=\frac{1}{2}|\vec{r}_1\times\vec{r}_3.|$$

A property of the orbit is that all the vectors $\vec{r}_i$ must lie in the same plane. Following Gauss, the equation of the orbit plane is given as

$$[2,3]\vec{r}_1-[1,3]\vec{r}_2+[1,2]\vec{r}_3=0. \quad (9)$$

Inserting equation (6) into equation (9) gives the result

$$[2,3]\hat{O}_1\rho_1-[1,3]\hat{O}_2\rho_2+[1,2]\hat{O}_3\rho_3=[2,3]\vec{S}_1-[1,3]\vec{S}_2+[1,2]\vec{S}_3. \quad (10)$$

The formal solution of (10) for the variables ρ_1,ρ_2,ρ_3 is obtained by Cramer's rule. The solution for ρ_1 is

$$\rho_1=\left(\Delta\frac{[2,3]}{[1,3]}\right)^{-1}\left(\frac{[2,3]}{[1,3]}C_1-C_2+\frac{[1,2]}{[1,3]}C_3\right), \quad (11)$$

where

$$C_1=\begin{vmatrix}X_1 & \lambda_2 & \lambda_3\\ Y_1 & \mu_2 & \mu_3\\ Z_1 & \nu_2 & \nu_3\end{vmatrix},\quad C_2=\begin{vmatrix}X_2 & \lambda_2 & \lambda_3\\ Y_2 & \mu_2 & \mu_3\\ Z_2 & \nu_2 & \nu_3\end{vmatrix},\quad C_3=\begin{vmatrix}X_3 & \lambda_2 & \lambda_3\\ Y_3 & \mu_2 & \mu_3\\ Z_3 & \nu_2 & \nu_3\end{vmatrix},\quad \Delta=\begin{vmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\\ \nu_1 & \nu_2 & \nu_3\end{vmatrix}. \quad (12)$$

The solution for ρ_2 is

$$\rho_2=-(\Delta)^{-1}\left(\frac{[2,3]}{[1,3]}D_1-D_2+\frac{[1,2]}{[1,3]}D_3\right). \quad (13)$$

where Δ is given in (12) and

$$D_1=\begin{vmatrix}\lambda_1 & X_1 & \lambda_3\\ \mu_1 & Y_1 & \mu_3\\ \nu_1 & Z_1 & \nu_3\end{vmatrix},\qquad D_2=\begin{vmatrix}\lambda_1 & X_2 & \lambda_3\\ \mu_1 & Y_2 & \mu_3\\ \nu_1 & Z_2 & \nu_3\end{vmatrix},\qquad D_3=\begin{vmatrix}\lambda_1 & X_3 & \lambda_3\\ \mu_1 & Y_3 & \mu_3\\ \nu_1 & Z_3 & \nu_3\end{vmatrix}. \tag{14}$$

The solution for ρ_3 is

$$\rho_3=\left(\Delta\frac{[1,2]}{[1,3]}\right)^{-1}\left(\frac{[2,3]}{[1,3]}E_1-E_2+\frac{[1,2]}{[1,3]}E_3\right), \tag{15}$$

where Δ is given in (12) and

$$E_1=\begin{vmatrix}\lambda_1 & \lambda_3 & X_1\\ \mu_1 & \mu_2 & Y_1\\ \nu_1 & \nu_2 & Z_1\end{vmatrix},\qquad E_2=\begin{vmatrix}\lambda_1 & \lambda_3 & X_2\\ \mu_1 & \mu_2 & Y_2\\ \nu_1 & \nu_2 & Z_2\end{vmatrix},\qquad E_3=\begin{vmatrix}\lambda_1 & \lambda_3 & X_3\\ \mu_1 & \mu_2 & Y_3\\ \nu_1 & \nu_2 & Z_3\end{vmatrix}. \tag{16}$$

The variables C_i, D_i, E_i (where i=1,2,3) and Δ are all *determinants*, which for the 3×3 matrices used here, can be evaluated using the general formula

$$\begin{vmatrix}d_1 & d_4 & d_7\\ d_2 & d_5 & d_8\\ d_3 & d_6 & d_9\end{vmatrix}=d_1(d_5d_9-d_6d_8)-d_2(d_4d_9-d_6d_7)+d_3(d_4d_8-d_5d_7). \tag{17}$$

All the information needed to calculate the determinants is known, since they are based on the observed comet positions and the known positions of the Sun. However, the areal ratios $[2,3]/[1,3]$ and $[1,2]/[1,3]$ are not known at this stage.

Gauss derived the following expressions for the areal ratios:

$$\frac{[2,3]}{[1,3]}=\frac{g_3}{f_1g_3-f_3g_1}$$
$$\frac{[1,2]}{[1,3]}=\frac{-g_1}{f_1g_3-f_3g_1} \tag{18}$$

where f_n and g_n are functions defined in the following way.

Firstly, the time variables τ_1, τ_3, τ_3 are defined where

$$\begin{aligned}\tau_1&=k(t_3-t_2),\\ \tau_2&=k(t_3-t_1),\\ \tau_3&=k(t_2-t_1),\end{aligned} \tag{19}$$

where k is Gauss's constant for the Solar System[2], formally given by the relation

$$k^2=GM, \tag{20}$$

in which G is the universal gravitation constant and M is the mass of the Sun.

Next the dynamical variables u_2, p_2, q_2 are defined where

2 Gauss's constant for the Solar System has the value k=0.01720209895 (a.u.)$^{3/2}$/day.

$$u_2 = \frac{1}{r_2^3},$$
$$p_2 = \vec{r}_2 \cdot \vec{v}_2 = r_2 \dot{r}_2, \tag{21}$$
$$q_2 = \frac{v_2^2}{r_2^2} - u_2.$$

Then the f_n and g_n functions (where n is 1 or 3) can be expanded in terms of the parameters u_2, p_2, q_2 as follows

$$f_1 = 1 - \frac{1}{2} u_2 \tau_3^2 - \frac{1}{2} u_2 p_2 \tau_3^3 + \frac{1}{24} u_2 (u_2 - 15 p_2^2 + 3 q_2) \tau_3^4 - \frac{1}{8} u_2 p_2 (7 p_2^2 - u_2 - 3 q_2) \tau_3^5 \ldots,$$
$$f_3 = 1 - \frac{1}{2} u_2 \tau_1^2 + \frac{1}{2} u_2 p_2 \tau_1^3 + \frac{1}{24} u_2 (u_2 - 15 p_2^2 + 3 q_2) \tau_1^4 + \frac{1}{8} u_2 p_2 (7 p_2^2 - u_2 - 3 q_2) \tau_1^5 \ldots, \tag{22}$$
$$g_1 = -\tau_3 + \frac{1}{6} u_2 \tau_3^3 + \frac{1}{4} u_2 p_2 \tau_3^4 - \frac{1}{120} u_2 (u_2 - 45 p_2^2 + 9 q_2) \tau_3^5 \ldots,$$
$$g_3 = \tau_1 - \frac{1}{6} u_2 \tau_1^3 + \frac{1}{4} u_2 p_2 \tau_1^4 + \frac{1}{120} u_2 (u_2 - 45 p_2^2 + 9 q_2) \tau_1^5 \ldots.$$

(Higher order expansions are available.)

In order to first obtain a preliminary orbit an attempt is made to obtain a good approximation for the value of the variable ρ_2, which corresponds to the central datum of the three comet observations (see [1-3]).

To begin with, the expansions (22) are truncated at order τ^3 and variable p_2 is assumed zero which gives

$$f_1 = 1 - \frac{1}{2} u_2 \tau_3^2 \ldots,$$
$$f_3 = 1 - \frac{1}{2} u_2 \tau_1^2 \ldots, \tag{23}$$
$$g_1 = -\tau_3 + \frac{1}{6} u_2 \tau_3^3 \ldots,$$
$$g_3 = \tau_1 - \frac{1}{6} u_2 \tau_1^3 \ldots.$$

Then, with the aid of equations (18), (19), (21) and (23), equation (13) can be written as

$$\rho_2 = A + \frac{B}{r_2^3}, \tag{24}$$

where

$$A = -(\Delta)^{-1} \left(\frac{\tau_1}{\tau_2} D_1 - D_2 + \frac{\tau_3}{\tau_2} D_3 \right) \tag{25}$$

and

$$B = -(6\Delta)^{-1} \left(D_1 \frac{\tau_1}{\tau_2} (\tau_2^2 - \tau_1^2) + D_3 \frac{\tau_3}{\tau_2} (\tau_2^2 - \tau_3^2) \right), \tag{26}$$

where terms of order higher than τ^3 have been discarded.

From the Sun-Earth-comet triangle (see Figure 1) the following relation is obtained using the cosine rule

$$r_2^2 = \rho_2^2 + S_2^2 - 2\rho_2 \hat{O}_2 \cdot \vec{S}_2, \tag{27}$$

which is required to calculate the magnitude of $\vec{r}_2$ once ρ_2 is known.

The procedure to obtain a *preliminary orbit* can now be described:

1. Guess an initial value of r_2.
2. Calculate A from (25) and B from (26) and use them with r_2 in equation (24) to estimate ρ_2.
3. Insert ρ_2 into (27) to get an update of r_2.
4. Repeat steps 2 to 3 until r_2 converges.
5. Insert r_2 into (21) to calculate u_2 and set $p_2 = 0$ and $q_2 = -u_2$.
6. Calculate $f_{1,} f_{3,} g_{1,} g_3$ using the expansions (22) then use (18) to compute the ratios $[2,3]/[1,3]$ and $[1,2]/[1,3]$.
7. Solve equations (11) and (15) to obtain ρ_1 and ρ_3 respectively.
8. Use equation (6) to obtain $\vec{r}_1, \vec{r}_2, \vec{r}_3$.
9. This completes the solution for the preliminary orbit.

The preliminary orbit is good enough for a first correction to the observation times for the time of light travel. The corrections take the form

$$\begin{aligned} \Delta t_i &= \rho_i / c, \\ t_i' &= t_i - \Delta t_i, \end{aligned} \quad \text{where } i = 1,2,3 \tag{28}$$

where c is the velocity of light[3]. The corrections are used to update the variables τ_1, τ_2, τ_3 as follows

$$\begin{aligned} \tau_1 &= k(t_3' - t_2'), \\ \tau_2 &= k(t_3' - t_1'), \\ \tau_3 &= k(t_2' - t_1'). \end{aligned} \tag{29}$$

The procedure for improving the preliminary orbit can now be described.

1. Using the current vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ and the light-corrected times τ_1, τ_2, τ_3 the velocity $\vec{v}_2$ at the second observed position is calculated using the formula
$$\vec{v}_2 = -\frac{\tau_1}{\tau_2 \tau_3} \vec{r}_1 + \frac{(\tau_1 - \tau_3)}{\tau_1 \tau_3} \vec{r}_2 + \frac{\tau_3}{\tau_1 \tau_2} \vec{r}_3. \tag{30}$$
2. Update u_2, p_2, q_2 using (21).
3. Update $f_{1,} f_{3,} g_{1,} g_3$ using the expansions (22) and obtain the ratios $[2,3]/[1,3]$ and $[1,2]/[1,3]$ using (18).
4. Update ρ_2 using (13) and use (27) to obtain new value of r_2.
5. Repeat steps 2 to 4 until ρ_2 and r_2 settle on fixed values.
6. Use equations (11) and (15) to obtain ρ_1 and ρ_3 respectively.
7. Apply light corrections (28) and (29).
8. Calculate $\vec{r}_1, \vec{r}_2, \vec{r}_3$ using equation (6).

10. Repeat steps 1 to 9 until ρ_1, ρ_2, ρ_3 no longer change.

The end result of this procedure is the converged set of vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ from which the standard orbital elements may be deduced using Gibbs' method. Alternatively, the

3 Which has the value c=173.1445988 a.u./day.

vectors $\vec{r}_2$ and $\vec{v}_2$ can be used for the same purpose. Both approaches are described in [5].

References

[1] *An Introduction to Celestial Mechanics*, F.R. Moulton, Dover Publications Inc. 1970.

[2] *Fundamentals of Astrodynamics*, R.R. Bate, D.D. Mueller, J.E. White, Dover Publications Inc. 1971.

[3] *Methods of Orbit Determination*, P.R. Escobal, Krieger Publishing Company Inc. 1976.

[4] *Practical Astronomy with Your Calculator*, P. Duffet-Smith, Cambridge University Press 1988.

[5] *Basic Orbit Mechanics*, W. Smith, available from: http://stargazy.weebly.com/essays-in-science.html