Newton’s Bucket and Telescope Mirrors

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Introduction

In Newton’s treatise, the Principia (1687), the author described an experiment he had devised to demonstrate the difference between relative and absolute motion. Characteristically, it was a simple experiment, but from it he drew deep insights into how our universe worked. His experiment consisted of a bucket of water suspended on a length of string tied to the handle. He twisted the bucket around many times until the string was extremely coiled and then let the bucket go. The bucket began to spin. Initially the water remained still and the surface stayed flat, but after a while the bucket passed its motion to the water which also began to spin. Gradually the water climbed the walls of the bucket and the surface curved into the shape of a parabola. Newton deduced from this that the water had acquired an absolute motion with respect to the universe, which was signalled by the change in the water surface and that this was physically different from the relative velocity of the bucket and the water at the start of the experiment.

Insightful as this experiment was, when I first came across it, my question was why the spinning surface of the water was a parabola and not some other shape? Curious to find out, I set about deriving the surface shape for myself, using the physical laws Newton himself had devised. I came up with two solutions which are presented here for readers who might be equally curious about such phenomena and also to point out an interesting connection with astronomical telescopes.

Method 1. The floating test particle

In this method we imagine a small particle, of mass m, floating on a liquid surface. If the surface is flat and undisturbed, the particle will remain stationary. This is easy to understand. The forces acting on it - the particle’s weight and the upthrust from the liquid - are equal and opposite and so cancel out one another (Archimedes c. 287–212 BC). A similar stasis also occurs in Newton’s bucket. When the spinning surface has achieved a stable shape, the particle holds a fixed position on the liquid surface. But the forces on the particle in this case are now subtly different.

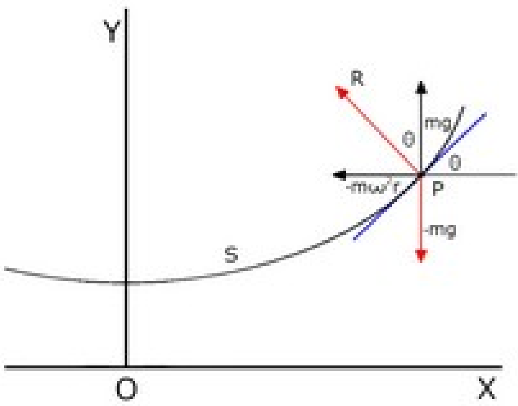

Figure 1. The particle P on the spinning surface

Figure 1 shows this phenomenon at a fixed instant in time. The liquid surface S is spinning about the vertical axis OY with an angular velocity ω. An axis OX is drawn perpendicular to the axis OY, such that the plane containing both axes also contains the particle at the time instant concerned. The two axes therefore define a frame of reference in which we may specify the position of particle P using coordinates (x, y), where x is the particle's radial distance from the axis of spin, and y is the vertical position of the particle. The point O in the figure is the origin of the reference frame at coordinates $(0,0)$.

The particle P in this circumstance is subject to the same two physical forces as before (shown in red in Figure 1). The first is its own weight $-mg$ (where m is the particle mass and g is the acceleration of gravity), which acts vertically downward as before. The second force is the upthrust R from the liquid surface, which acts in a direction perpendicular to the liquid surface. However, it no longer acts wholly in the vertical direction. In fact the force R may be resolved into two components: R_y acting vertically in opposition to the particle's weight, and R_x acting horizontally and providing the *centripetal force,* so that the two components acting together hold the particle in position on the surface. Therefore the components (which are drawn in black in Figure 1) can be written as

$$R_x = -m\omega^2 x, \quad \text{and} \quad R_y = mg. \tag{1}$$

In Figure 1 we have defined an angle θ between the force R and R_y, so that we may write the ratio of R_y and R_x as

$$\frac{R_y}{R_x} = -\frac{g}{\omega^2 x} = \tan(\theta + \pi/2), \tag{2}$$

where $\theta + \pi/2$ is the angular direction of the vector R in the reference frame. However, from standard trigonometry

$$\tan(\theta + \pi/2) = -\frac{1}{\tan\theta}. \tag{3}$$

So from equations (2) and (3) we can see that

$$\tan\theta = \frac{\omega^2 x}{g}. \tag{4}$$

Now, if the tangent to the curve S is drawn at P (the blue line in Figure 1), this is necessarily perpendicular to R and from that it follows that the tangent is at an angle θ to the horizontal. So from (4) we can write for curve S

$$\tan\theta = \frac{dy}{dx} = \frac{\omega^2 x}{g}. \tag{5}$$

Integrating equation (5) over x gives

$$y=\frac{\omega^2}{2g}x^2+c, \qquad (6)$$

where c is an arbitrary constant that depends on where the origin O of the coordinates is set. Equation (6) is of course a parabola. So the curve defining the liquid surface in Newton's bucket is indeed parabolic.

Method 2. Using the liquid pressure/depth relation

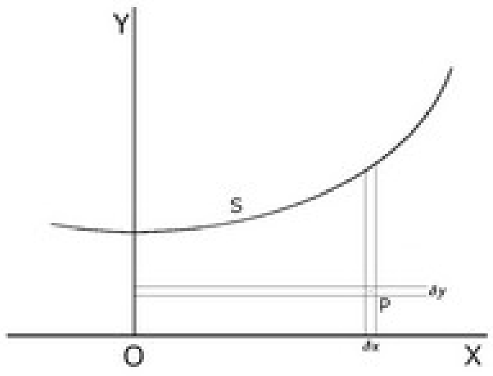

Figure 2. Calculating the pressure at position P beneath the liquid surface.

This approach is based on the well known expression for the pressure p in a liquid of density ρ at a depth h below the surface, which is

$$p=\rho h g. \qquad (7)$$

We can use this to calculate the shape of the surface as follows.

Figure 2 represents the spinning surface, frozen at an instant in time. Axes OX and OY define the coordinate frame, as before. The point P below the liquid surface has coordinates (x, y) and is the centre of a small cubic volume of liquid with dimensions $\delta x, \delta y, \delta z$. The volume δV of this cube is given by the formula

$$\delta V=\delta x\,\delta y\,\delta z \qquad (8)$$

and the mass δm of this volume of liquid is obtained from the density ρ and the volume conjointly as

$$\delta m=\rho\delta V=\rho\,\delta x\,\delta y\,\delta z. \qquad (9)$$

Since the liquid is static with respect to the rotating frame, this mass rotates in a circle around the spin axis OY and must be held in place by a centripetal force δf of magnitude $\delta m\omega^2 x$, which equation (9) allows us to write as

$$\delta f=\rho\omega^2 x\,\delta x\,\delta y\,\delta z \qquad (10)$$

The origin of this force is the pressure difference between the parallel vertical faces of the cube at the positions $x+\delta x/2$ and $x-\delta x/2$. These two faces having the same area: $\delta y \times \delta z$. According to equation (7) the pressure difference δp between the faces arises from the difference in the surface heights at positions $x-\delta x/2$, and $x+\delta x/2$, which we write as

$$\delta p = \rho g (y(x+\delta x/2) - y(x-\delta x/2)). \tag{11}$$

The force δf holding the mass δm in place is the pressure δp multiplied by the area $\delta y \times \delta z$, which gives

$$\delta f = \rho g (y(x+\delta x) - y(x-\delta x)) \delta y \delta z. \tag{12}$$

Note that, since the height of the curve at $x+\delta x/2$ is greater than at $x-\delta x/2$ at P, it follows that this force acts towards the axis OY. This is the force identified above as the centripetal force (10), so from (10) and (12) it is apparent that

$$\frac{(y(x+\delta x/2) - y(x-\delta x/2))}{\delta x} = \frac{\omega^2}{g} x, \tag{13}$$

which in the limit as $\delta x \rightarrow 0$ becomes

$$\frac{dy}{dx} = \frac{\omega^2}{g} x. \tag{14}$$

This is the same as equation (5) and so integrates to the same result (6), thus proving once again the parabolic shape of the spinning surface.

The Parabola Focus and Directrix

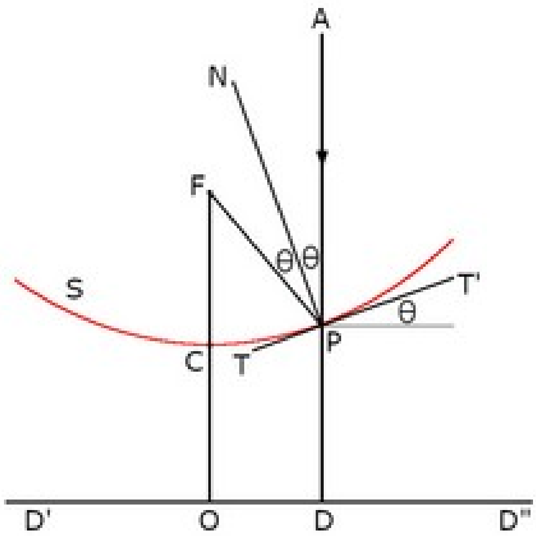

Figure 3. Properties of a parabola.

A parabola is formally a *conic section*, which is a curve for which the distance of every point P on the curve, from a fixed point (the *focus*), is in a fixed ratio to its perpendicular distance from a straight line (the *directrix*). This is expressed in the equation

$$e=\frac{d_f}{d_d}, \tag{15}$$

where d_f is the distance of the point P from the focus and d_d is its perpendicular distance from the straight line. The eccentricity e is a positive number, which for a parabola is $e=1$, for an ellipse is $e<1$ and for a hyperbola is $e>1$. For our purpose here we focus on a few properties of the parabola.

Consider Figure 3. The red curve S is a parabola, line DD′ is the (horizontal) directrix and F is the focus. Point P is on the parabola and the line PD is a perpendicular dropped to the directrix. It follows from Equation (15) that the distance PD equals FP since $e=1$. Likewise, the perpendicular line dropped from F to the directrix at O cuts the parabola at C so that lengths FC and CO are also equal. Point C is the closest point on the parabola to the focus and the line FO lies on the *principal axis* of the parabola.

Line TT′ is drawn to bisect the angle FPD into equal parts and is the tangent to the curve at P. This makes an angle θ to the horizontal. Line NP is drawn as a perpendicular to line TT′ at P and the line AP is drawn as a linear extension of line PD. It follow that angle APT′ is $\pi/2-\theta$, and since angle NPT′ is a right angle by construction, angle NPA must be equal to θ.

Angle APT′ is equal to angle TPD, as corresponding angles, and angle TPD is equal to angle TPF by construction. So angles TPF and APT′ are equal. It therefore follows that angle NPF is also equal to θ and thus line NP bisects the angle APF into equal parts. From this result we can therefore say that a light ray travelling parallel to the principal axis and striking a parabolic mirror of this form at P will, by the laws of reflection, be reflected towards the focus. This is true for all points P. This is the basis for using parabolic mirrors in telescopes.

If we have an equation of a parabola in the form

$$y=Ax^2, \tag{16}$$

this curve passes through the origin at $(0,0)$ which corresponds to the lowest point on the curve. For this equation, the focus is perpendicularly above the origin. Both the origin and focus lie on the principal axis.

We can determine the focus and directrix as follows. For a point $P=(x',y')$ on the parabola (with $x'\neq 0$) we have

$$y'=A(x')^2, \tag{17}$$

and by differentiation

$$\tan\theta=2Ax'. \tag{18}$$

By inspection of Figure 3 it can be deduced that the height f of the focus above the origin at $(0,0)$ is given by

$$f = y' + \frac{x'}{\tan 2\theta}, \quad \text{i.e.} \quad f = y' + x' \frac{1-\tan^2\theta}{2\tan\theta}. \tag{19}$$

Substituting for y' and $\tan\theta$ using equations (17) and (18) and cancelling terms gives the result

$$f = \frac{1}{4A}. \tag{20}$$

For a parabola the directrix is the same distance *below* the origin i.e. at $-f$.

In the case where $A = \omega^2/2g$ we would obtain

$$f = \frac{g}{2\omega^2}. \tag{21}$$

This parameter is the focal length of a parabolic mirror made by spinning.

Application to Telescope Mirrors

The relevance of Newton's bucket to telescopes is obvious to any astronomer. The common Newtonian telescopes use a mirror to gather and focus the light and the required shape for that mirror is parabolic. So there naturally arises a question of how to use the idea of a spinning liquid surface in a real telescope.

It seems unlikely that anyone would make, use or even want a telescope that was in any way spinning, particularly when it is realised that such a telescope would be confined to looking straight up into the sky! This limitation could, however, be offset somewhat by having the viewer work off the principal axis, as Herschel did with his 40 foot telescope. So it turns out that somebody has actually done this. The Large Zenith Telescope that once existed in British Columbia in Canada, sported a 6 metre mirror made from liquid mercury! However, it was largely an experimental device and not used for practical astronomy.

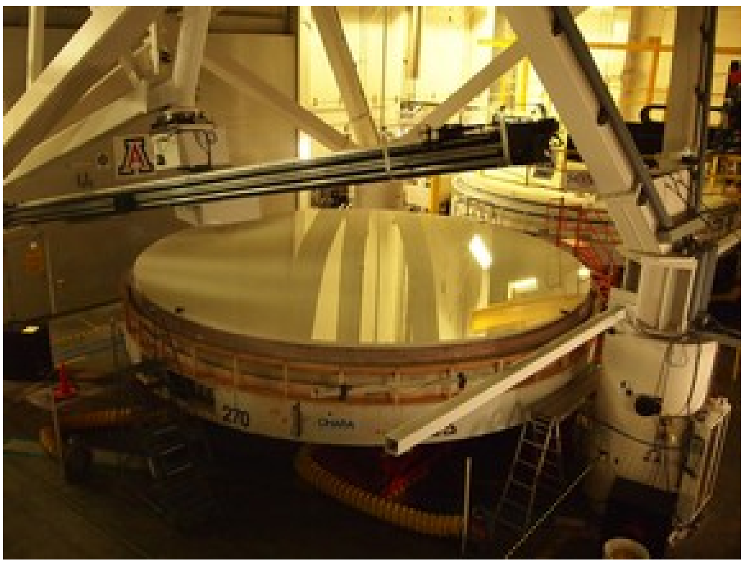

Figure 4. Polishing an astronomical mirror in the SOML.

A more practical application of the idea is in the manufacture of glass telescope mirrors, as is done at the Steward Observatory Mirror Laboratory (SOML), beneath the Wildcats football stadium(!) in Tucson, Arizona. There a vat of molten glass is slowly rotated in a circular mould as it cools to a solid glass parabola. Mirrors up to 8.4 metres in diameter are manufactured and these have been installed in some of the largest astronomical telescopes in the world. If an observatory boasts a telescope with a mirror 8 metres wide, there's a good chance it's a mirror made in the SOML. You may ask why stop at 8.4 meters? Because anything larger could not get out of the building! It's housed in a football stadium after all, not a purpose built factory.

The great advantage of spinning glass to make mirrors is that it greatly reduces the amount of time it requires to polish the mirror to optical perfection. Prior to this, a flat 'blank' mirror had to be ground for an inordinate amount of time to a parabolic shape and in the process it was necessary to remove a huge amount of glass (see below). Now that a large mirror can be shaped accurately by spinning, this is no longer so tedious. However, a measure of fine polishing is still required to perfect the surface (see Figure 4). All this was the brain child of Roger Angel, a British born Oxford graduate and presumably, a Newton acolyte.

It is interesting to attempt to design a telescope mirror, based on what we know. Suppose we want an 8 metre diameter mirror with a focal length of 80 meters, which would make it an F10 mirror, suitable for a large Cassegrain telescope.

The equation for this mirror surface is equation (6) with $c=0$ i.e.

$$y=\frac{\omega^2}{2g}x^2. \tag{22}$$

The key engineering parameter we need to know is the rate of rotation of the vat holding the molten glass to obtain the required focal length. Rearranging equation (21) gives us

$$\omega^2=\frac{g}{2f}. \tag{23}$$

In metric units, $g=9.81\ m/s^2$ and we have $f=80\ m$. The result from equation (23) is $\omega=0.24761\ rad/s$, or $2.36454\ rpm$. This is fortunately a low speed of revolution, given the mass of glass being spun (see below).

A useful alternative to the form (22) is obtained by replacing ω^2 in (22) using (23) to express the equation of the surface purely in terms of the focal length:

$$y=\frac{x^2}{4f}. \tag{24}$$

What amount of glass do we require? A useful parameter for assessing this is the vertical height h from the centre of the mirror to the outer rim. This is easily determined. Writing the diameter of the mirror as D, the height of the rim is obtained from (24), it is:

$$h = \frac{D^2}{16f}. \tag{25}$$

Thus for our $8m$ diameter mirror, we obtain $h = 0.05\ m$. From this we may work on the premiss that the thickness T of the slab of molten glass required is of the order of $2h$, which would ensure that the parabolic surface could not reach down to the base of the slab and compromise the lower surface. Other considerations may also apply, such as giving the slab extra thickness for rigidity or a honeycomb structure to reduce weight. Our calculation is therefore merely indicative. Based on our calculation of h, we will set $T = 0.1\ m$. The volume of glass required is therefore

$$V = \frac{\pi}{4} T D^2. \tag{26}$$

The calculated volume is $V = 5.0265\ m^3$. Given the density of glass ρ of, say, $2500\ kg/m^3$ (this varies with the composition of the glass) we calculate the mass M of the mirror using

$$M = \rho V, \tag{27}$$

which gives a mass of $12566\ kg$ or 12.566 metric tonnes.

A useful quantity to know is the volume V_0 of the bowl of the mirror, which we may think of as the volume of glass that would have to be ground away to make the mirror from a cylindrical slab of glass. We can calculate this by subtracting the volume of glass remaining beneath the surface from a circular slab of diameter D and height h. Using equation (24) we obtain the following equation

$$V_0 = \frac{\pi}{4} D^2 h - \frac{\pi}{2f} \int_{x=0}^{D/2} x^3\, dx, \tag{28}$$

which becomes on integration

$$V_0 = \frac{\pi}{4} D^2 \left(h - \frac{D^2}{32f} \right). \tag{29}$$

For a mirror with our specification we obtain the result $V_0 = 1.2566\ m^3$, which implies the removal of $3{,}142$ metric tonnes of glass. So grinding out the surface mirror from a blank disk of glass would indeed be a formidable task. A further advantage of knowing V_0 is that we now know the mirror needs $3{,}142$ metric tonnes less glass that we originally estimated.