A Note on Refraction

W. Smith

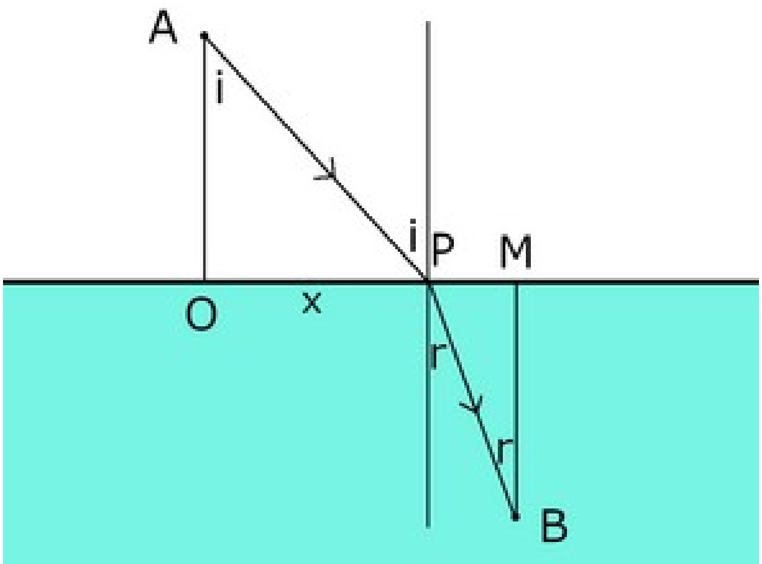

Figure 1.Refraction at an Interface

Figure 1 is a diagram tracing a ray of light from a point A above an interface to a point B below the interface, where the refractive index μ is higher than above the interface. From Snell's law of refraction it is known that

$$\mu = \sin i / \sin r \, , \tag{1}$$

where i and r are the angles of incidence and refraction respectively.

Travelling from A the light ray strikes the interface at point P and then proceeds along a different straight line to B. We will show that the path taken from A to B is the quickest path the light can follow.

Point P is at a distance x from the point O, which is perpendicularly below A. Point M on the interface is perpendicularly above point B and the distance from O to M is specified as L. It follows that

$$\tan i = \frac{x}{AO} \quad \text{and} \quad \tan r = \frac{(L-x)}{BM} \; . \tag{2}$$

Also, from Figure 1 we see that

$$\frac{AP}{AO} = \frac{1}{\cos i} \quad \text{and} \quad \frac{PB}{BM} = \frac{1}{\cos r} . \tag{3}$$

From Pythagoras, the path AP has a length

$$AP = (AO^2 + x^2)^{1/2}, \tag{4}$$

and the path PB has a length

$$PB = (BM^2 + (L - x)^2)^{1/2}. \tag{5}$$

Therefore the time t_{AB} for light to travel the path $AB = AP + PB$ is given by

$$t_{AB} = \frac{AP}{c_{AP}} + \frac{PB}{c_{PB}}, \tag{6}$$

where c_{AP} is the speed of light along path AP and c_{PB} is the speed along path PB. (These two speeds are not necessarily the same.)

From relations (4) and (5) equation (6) becomes

$$t_{AB} = \frac{1}{c_{AP}}(AO^2 + x^2)^{1/2} + \frac{1}{c_{PB}}(BM^2 + (L - x)^2)^{1/2}. \tag{7}$$

To find the quickest path the light can follow from A to B, we must differentiate the time t_{AB} with respect the variable x. So (7) becomes

$$\frac{dt_{AB}}{dx} = \frac{x}{c_{AP}}(AO^2 + x^2)^{-1/2} - \frac{(L - x)}{c_{PB}}(BM^2 + (L - x)^2)^{-1/2} = 0. \tag{8}$$

Which holds for a maximum or minimum path. Taking the terms AO^2 and BM^2 outside the brackets in (8) gives

$$\frac{dt_{AB}}{dx} = \frac{x}{c_{AP} AO}\left(1 + \frac{x^2}{AO^2}\right)^{-1/2} - \frac{(L - x)}{c_{PB} BM}\left(1 + \frac{(L - x)^2}{BM^2}\right)^{-1/2} = 0. \tag{9}$$

Using the identities (2), equation (9) becomes

$$\frac{dt_{AB}}{dx} = \frac{\tan i}{c_{AP}}(1 + \tan^2 i)^{-1/2} - \frac{\tan r}{c_{PB}}(1 + \tan^2 r)^{-1/2} = 0. \tag{10}$$

Since for any angle α it is true that $sec^2 \alpha = 1 + \tan^2 \alpha$, equation (10) easily reduces to

$$\frac{dt_{AB}}{dx} = \frac{\sin i}{c_{AP}} - \frac{\sin r}{c_{PB}} = 0. \tag{11}$$

From which it follows that

$$\frac{c_{AP}}{c_{PB}} = \frac{\sin i}{\sin r} = \mu. \tag{12}$$

Thus it transpires that Snell’s law is compatible with the requirement that light follows the *extrema*l path between A and B. It is necessarily the shortest path in this case since it is easy to construct an alternative path that is slower than the one taken. It also shows that the refractive index is the ratio is the velocity of light above the interface to its velocity below the interface – which interestingly contradicts Newton's hypothesis that light in a medium with $\mu>1$ travels faster than in a vacuum.

What this analysis shows is that, with regard to the question of the path taken by the light ray between points A and B, the assumption that it follows the minimum time path from A to B turns out to agree with the refraction law derived experimentally by Snell. It begs the question of how a photon departing from point A is able to choose such a path, which is only apparent after the photon completes its journey. Such *a priori* information seems mystical.

It is easy to show however, that for every photon (obeying Snell's law) passing through point A from somewhere else only those photons that follow the minimal time path we have found above will actually pass through point B. The rest will follow completely different paths. So from the perspective of physics, the actual path is the only one that can connect points A and B. The fact that it is also minimal in the sense revealed by the above analysis is perhaps a physical property of space and time and the photon is *compelled* to take the path it does.

W. Smith 2023.