On the Shape of the Earth

W. Smith

The fact that the Earth is a sphere is well known. It is perhaps less well known that the Earth is actually not a perfect sphere and can best be described as an oblate spheroid. This means that a cross section of the Earth, taken from pole to pole, is an ellipse, not a circle. According to Wikipedia the distance between the North and South poles (representing the shortest diameter of the ellipse) is 12,713.8 km, while the corresponding distance at the equator (representing the longest diameter of the ellipse) is 12,756.3 km. So the polar diameter 42.5 km less than the equatorial diameter. Why is this?

The famous English scientist Isaac Newton (1643-1727) was the first person to provide an explanation. It is the result of the balance between gravity, which tends to pull the mass of the Earth into a sphere, and the rotation of the Earth, which tends to push the mass of the Earth away from the axis of rotation. He demonstrated this in the following manner.

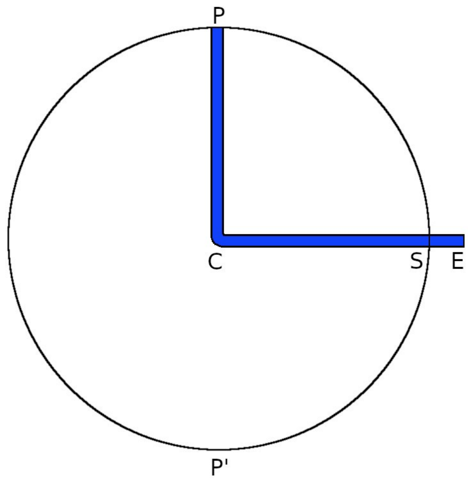

Figure 1: Axial Channels of Earth

In Figure 1 we have a representation of Earth in which water channels have been dug to the centre of the Earth at C. The first is from the North pole P to C, and the second is from a point S on the equator to C. At the point S the channel is further extended vertically above the ground to a point E so that, if necessary, a column of water can be accommodated should the water level rise due to the rotation of the Earth.

It is assumed that the Earth is a perfect sphere, which like the real Earth is rotating about the polar axis PP′ at a rate ω of 360° in 23 hours, 56 minutes and 4 seconds (so ω is $7.292125\text{x}10^{-5}$ rad/s). We also assume that the Earth has a diameter of 12,713.8 km, which is the actual polar diameter of the Earth. We now fill the water channel by pouring water down from the North pole P and keep on pouring until the water level at P is flush with the surface of the Earth there. We now seek to determine the water level at the equator surface S.

Since the water is at equilibrium when the channel is full, we can say that the pressure at the centre C can be calculated either from the the column of water in channel CP or from the column in channel CE, the result will be the same. However the column of water in CP will have a height R=6.3569x10^6 m, which equals half the diameter PP', while the height of the water in CE will be some value H, where H $\neq$ R, according to our expectations. We will derive expressions for the pressure in each channel assuming the water is of uniform density ρ throughout.

In channel CP, a volume δV of water at a height r (measured from the centre C), makes a contribution δP of to the pressure at centre C given by

$$\delta P_{CP} = \frac{G \rho M(r)}{\sigma r^2} \delta V, \tag{1}$$

in which G is the universal gravitational constant (6.6726x10^{-11} in MKS units), σ is the cross-sectional area of the channel and $M(r)$ is the mass of the Earth from the centre to the radius r. Since we also have $\delta V = \sigma \delta r,$ equation (1) becomes the integral

$$P_{CP} = G \rho \int_0^R \frac{M(r)}{r^2} dr. \tag{2}$$

If we assume a uniform Earth density $\rho_E,$ then $M(r) = 4 \pi \rho_E r^3 / 3,$ and equation (2) becomes

$$P_{CP} = \frac{4}{3} \pi G \rho \rho_E \int_0^R r\, dr = \frac{4}{3} \pi G \rho \rho_E \frac{R^2}{2} = \frac{G \rho M_E}{2R}, \tag{3}$$

where $M_E = M(R)$ is the mass of the Earth (5.9723x10^{24} kg).

The pressure calculation for the channel CE includes precisely the same contribution from gravity as given in result (3), but it differs two respects. Firstly, in addition to the gravitational force acting on the water column, there is also a centrifugal force given as $\rho \omega^2 r \delta V$ which opposes gravity by acting in the opposite direction. Overall the centrifugal force contributes the (negative) pressure term

$$P_{CE}^{a} = -\rho \omega^2 \int_0^H r\, dr = -\frac{1}{2} \rho \omega^2 H^2. \tag{4}$$

Note that the upper limit of this integral is the distance H, not R.

Secondly, there is an additional pressure contribution arising from the gravitational force acting on the (supposed) additional column of water in the channel SE, which is given as

$$P_{CE}^{b} = G\rho M_E \int_R^H \frac{dr}{r^2} = G\rho M_E \left(R^{-1} - H^{-1}\right). \tag{5}$$

Taking all contributions into account, the pressure equation for channel CE is

$$P_{CE} = \frac{G\rho M_E}{2R} - \frac{\rho\omega^2 H^2}{2} + G\rho M_E \left(R^{-1} - H^{-1}\right). \tag{6}$$

Now we require that $P_{CP} = P_{CE}$, so it follows from comparing equations (3) and (6) that

$$\frac{-\rho\omega^2 H^2}{2} + G\rho M_E \left(R^{-1} - H^{-1}\right) = 0. \tag{7}$$

Cancelling common terms and rearranging gives

$$\frac{G M_E (H-R)}{HR} = \frac{\omega^2 H^2}{2}. \tag{8}$$

Since we are expecting the difference $h = H - R$ to be small then $H \sim R$, and we can approximate (8) as

$$h \sim \frac{\omega^2 R^4}{2 G M_E}, \tag{9}$$

in which h represents the height of the water column in channel SE. Note that this result is independent of the density of water, so we could have used some other liquid (perhaps molten lava!) instead. We may now Calculate h using previously defined values of ω, R, G and M_E.

$$h \sim \frac{(7.292125 \times 10^{-5})^2 \times (6.3569 \times 10^6)^4}{2 \times 6.6726 \times 10^{-11} \times 5.9723 \times 10^{24}}. \tag{10}$$

From this we find h~10.895 km. This is the difference in height between the polar and equatorial water columns. If we assume the Earth was once molten liquid which solidified into its present shape, the distance h corresponds to half the difference in the respective diameters of the Earth. So the difference in diameters we obtain for this calculation is 21.79 km. This compares with the actual difference which is 42.5 km. Given that our approach makes some assumptions about the physical constitution of the Earth and that Earth's actual history is uncertain (did it really solidify from a liquid state?), plus our neglect of the influence of the Moon, this result is remarkably close.

It is interesting to perform this calculation with respect to Jupiter, which has a mass of $1.8982 \text{x} 10^{27}$ kg, rotational velocity $1.7585 \text{x} 10^{-4}$ rad/s, polar diameter 133,708 km and equatorial diameter 142,948 km. Putting these numbers into the formula (10) returns a value h~2,438.5 km or a difference in diameters of 4,877 km against a real difference of 9,240 km, so again the calculation reveals the right order of magnitude and about half the true value. Overall this is an impressive result.