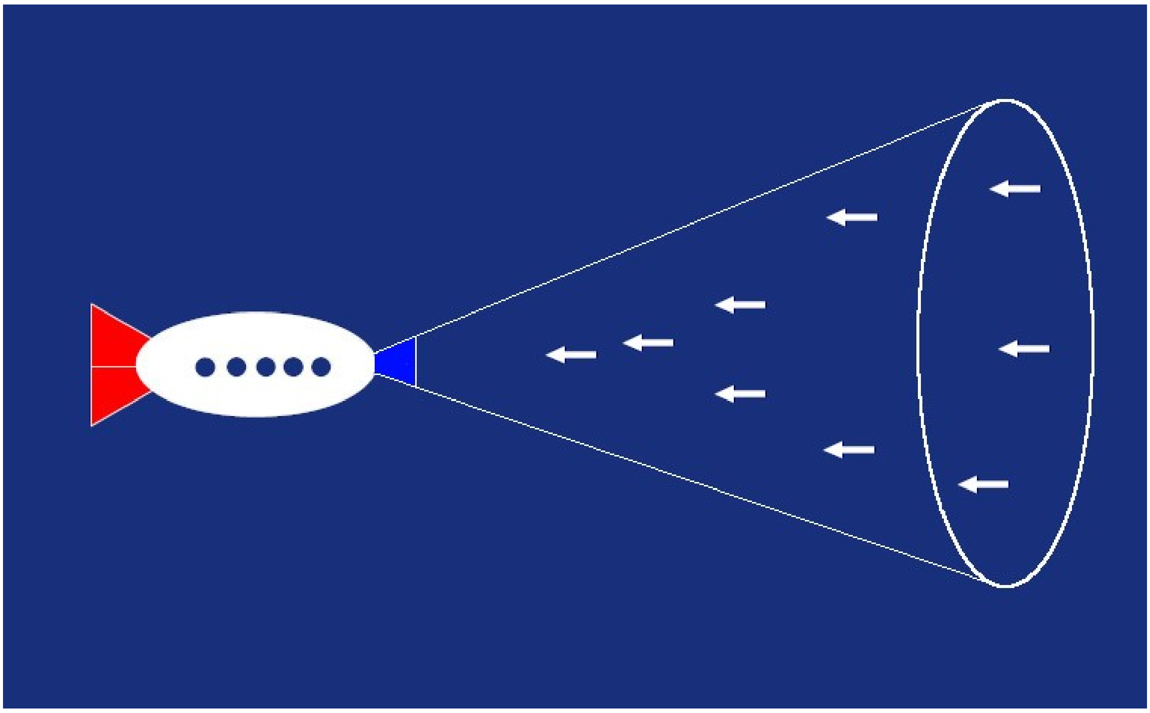
Star Ship Relativity

A monograph on relativity
and interstellar space travel

Bill Smith

To James, who loved the
idea of space travel.



Acknowledgement

In writing this monograph I have been influenced by many previous writers whose books I have read down the years. In truth there are too many to name, or even recall precisely, but I wish to acknowledge my debt to them all, named and unnamed. Some however, have been particularly influential in that the things they wrote about have found re-expression here. I realise this does not necessarily mean they are the originators of the ideas I have taken on board, but they brought them to my attention and that is enough for me to commend them. From the book *Relativity and Common Sense* by Hermann Bondi I learned the value of simple space-time diagrams in explaining relativity and also that using light units can be revealing when deriving the Lorentz transformation. My appreciation of the power of space-time diagrams was amplified by the book *A Very Special Relativity* by Sander Bais. The book *Relativity Physics* by William H. McCrea and *An Introduction to the Special Theory of Relativity* by Robert Katz both managed to teach me the mathematics of relativity in a way I found non-intimidating and it was in the book *Principles of Cosmology and Gravitation* by Michael Berry that I first learned that the space-time interval offers a neat way into the special theory of relativity and indeed provides an entry point into the general theory as well! All these books delighted me despite (in some cases) being several decades old, and I am grateful tor them. Needless to say, the imperfect scientist that I am, asks forgiveness if I have presented their ideas incorrectly. Any mistakes are surely mine, not theirs.

Preface

Given that there are already many excellent books on Einstein's theory of relativity, it begs the question: why should there be another – albeit a mere monograph? My answer is that Einstein's theory is intellectually challenging and since different people learn things by following different paths another, alternative path is self evidently a good thing. My own path has taken me through many books on the subject. I took my understanding wherever I could get it as I tried to build a coherent picture of the subject. It is a path that has teased and excited me in turns and frankly, at times, left me astonished at Einstein's audacity – how could he even think that? The result of my journey is presented here in these pages. In keeping with modern times it is cast in terms of `star ships' capable of interstellar flight, rather than trams running through the streets of Zürich. This fiction allows me to present some of the more extreme aspects of relativity theory in a way that shows why the theory is so challenging to everyday experience. There may be some lessons for the future too, as I raise some serious questions about the possibility of interstellar space flight.

I hope others will find my monograph a useful assistant on their own path of discovery.

1. The Universe from a Star Ship

Imagine you are in a star ship far out in space - beyond the solar system, beyond even the Milky Way, somewhere between the distant galaxies (Figure 1), which appear as faint, luminous smudges through the view ports of the ship. The interstellar drive is dormant and the ship cruises at a constant velocity through the void. Gravity is virtually absent out here, being far too weak to detect even by the most sensitive instruments. In these circumstances it will seem to you that you are at rest, with no motion whatsoever, as you seemingly hang suspended between the surrounding galaxies. This sense of being unable to determine your own motion when not subject to forces of any kind was well known to the great Italian scientist Galileo Galilei, who proposed that it was a property of the universe, and not just an illusion[1]. We shall call this Galileo's Principle. It turns out to be a valuable idea.

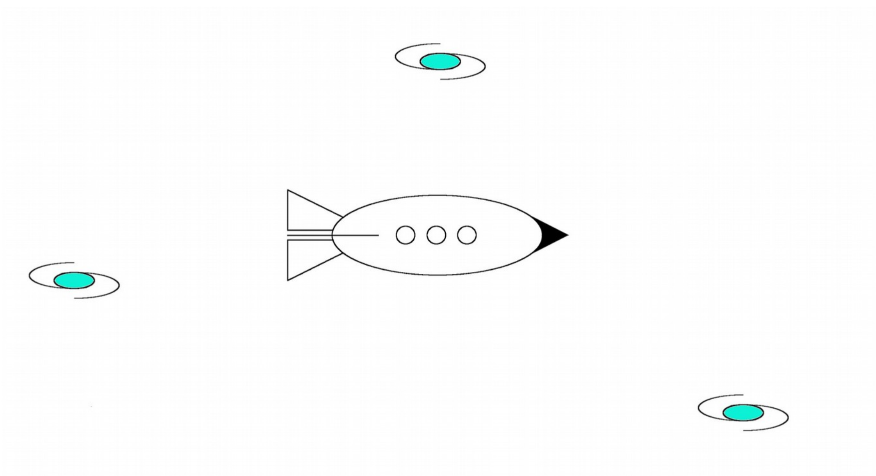


Figure 1. Our star Ship

In this circumstance a star ship is a close approximation to something that physicists call an *inertial frame of reference,* and it is a place from which we can describe the universe around us. In effect, we can assume we are stationary in space and define the position of every other object we observe with respect to our own *local* reference frame. Stationary objects are identified as holding a fixed position in our reference frame, while moving objects are those that change their positions with time. This presents a convenient and practical framework in which to describe everything we observe in the universe.

It is clear, however, that some other star ship, cruising through space in a similar manner, could equally consider itself to be stationary and describe the universe in terms of its own local frame. In fact *any* star ship cruising at constant velocity, far from gravitational influences, is a valid inertial frame of reference. This multiplicity of

1 Note however, that this observation does not hold for rotational motion, for it is all too apparent when we are being rotated: even very distant galaxies would appear to change position and we would also *feel* the rotation as a centrifugal force. Rotational motion therefore is quite distinct from linear motion.

valid local frames of reference is the root of a fundamental question: all these frames are equivalent, but each measures the universe independently, so how do we reconcile measurements made in one frame with those made in another? And if the results are different, how can we be sure they describe the universe in the same way? This is the central question of the theory of relativity, which is the subject of this essay.

1. The Local Reference Frame

Before we tackle the theory of relativity, we should first become familiar with the properties of the local inertial frame, so that we understand what measurements we are talking about.

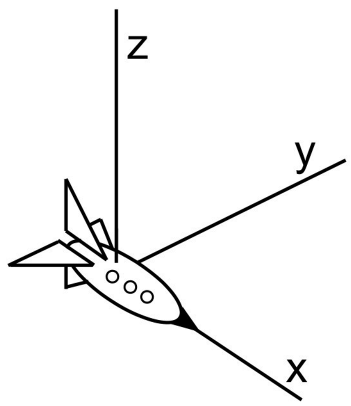


Figure 2. The Coordinate Axes of the Star Ship

In any given inertial frame of reference, such as a star ship, we can set up a coordinate system that allows us to state precisely the positions of objects in space. The actual coordinate system can be chosen arbitrarily, but once chosen must be strictly adhered to forever. Such a coordinate system is shown in Figure 2, where we have a simple set of mutually perpendicular, right handed[2] Cartesian axes centred on the ship. In this arrangement the centre of mass of the ship is at the so called *origin* of the x, y and z coordinates, which we record as a triplet of numbers $\vec{O}=(0,0,0)$, which is the so-called *vector notation* for such a triplet. The x-axis is aligned along the supposed direction of motion of the ship, which is towards the front end. The y- and z-axes are then perpendicular to this and to each other. To be specific, we could say the y-axis points out through the side of the ship marked by some structural feature, such as an entrance hatch, just so we can agree on its direction, then the z-axis is drawn perpendicular to both x- and y-axes.

Once the axes have been defined we may quantify the position of any object in space against each of these axes in the manner familiar to anyone who has ever drawn a graph: from the object's position in space, a perpendicular line is drawn towards each of the three axes, meeting each of them at an angle of 90^o. Then we measure the distance along each axis from the coordinate origin $\vec{O}$ to where these perpendicular

2 Right handed coordinates: Hold up your right hand and arrange your forefinger, thumb and middle finger in mutually perpendicular directions. Your forefinger would then form the x-axis, your middle finger the y-axis and your thumb the z-axis.

lines meet each axis. These three measurements are the Cartesian coordinates and are written as the triplet of numbers $\vec{r}=(x,y,z)$, where $\vec{r}$ is the so-called *position vector* and x,y and z are the *components* (or *coordinates*) measured along the appropriate axis. Of course, in order to make a measurement of position we need some sort of ruler or measuring tape calibrated in standard units (e.g. metres), to make our measurements universally meaningful.

However, if a measurement pertains to a moving object, the position recorded is only valid at the time the observation is made, so it is also necessary to record the time of the observation if we are to specify it properly. Indeed, it is inevitable that many of the objects seen from the star ship will be moving. For this purpose we need a clock, that measures time in regular "ticks" with respect to some chosen time origin (defined as zero time), from which we can specify the time t at which the observation is made. As with distance, we need standard time units (e.g. seconds) to give universal meaning to the numbers we record. The combined set of four numbers (t,x,y,z) are collectively known as an *event*[3] – since they specify a place in time *and* space. A time ordered series of events relating to an object is known as the *history* of the object. Furthermore, if we could imagine a 4 dimensional Cartesian space for the coordinates (t,x,y,z), the full history of an object would constitute a path through that space called the *world line*.

Making measurements of time and position is the foundation of physics. We expresses what we observe as numbers, which we can process with mathematics and establish mathematical relations that we call physical laws. With these we can make testable predictions for further experimental verification. In this way we develop scientific theories of the universe. That is how science progresses. In general we are not just interested in time and distance, but other physical properties (or *variables*) as well, such as mass, velocity, force, electrical charge, magnetic strength and so on – in fact *anything* we can measure. So the relations we derive can be *multi-variate* and sometimes complicated. It is fair to say though, that the most fundamental variables are space and time. Therefore with our coordinate system, calibrated with rulers and clocks, we have a good basis for building the science of physics.

Now, as we have mentioned previously, *any* inertial reference frame is equivalent to the one shown in Figure 2 and every cruising star ship will have its own local reference frame. However, the equivalence of frames does not mean the same event yields the same components in different frames. Indeed it should be obvious that the different local frames cannot record things identically, since they have a different *perspective* of events. However we must assume that any differences observed will be easily explained once we take account of the differences between the frames themselves. This must be so if the laws of physics are to be same for all observers. This belief is the basis of relativity theory. In practical terms, it is necessary to define a *transformation*, which is a mathematical formula for converting the measurements obtained in one frame into the corresponding measurements obtained in another. A little thought tells us that the actual form of this transformation represents a fundamental property of the universe, since it enables the laws of physics to be

3 Here we could define an event vector $\vec{e}=(t,x,y,z)$ as the 4-dimensional equivalent of the position vector $\vec{r}$, but we refrain here because t is measured in different units from $x,y,$ and z and normally vector components all have the same units.

applied in the same way in all inertial frames of reference. Our job is to find the form of this transformation and ensure it preserves the universality of physical laws.

2. Science On Board a Star Ship

What is the nature of the science we would find applicable in the local frame of reference aboard our star ship? It would in fact be almost the same as the science we have on Earth. Chemistry and biology would be identical to Earth's and physics would differ only because gravity's influence is absent on the star ship and, in the absence of Earth's rotation, the stars would appear stationary in the heavens. In particular, Isaac Newton's laws of motion (which we discuss later) would be fully applicable as would the theory of electromagnetism of James Clerk Maxwell. Indeed from Maxwell's theory it follows that light would behave in the same predictable way as on Earth. One can even say that the laws of quantum mechanics would apply unchanged, which of course is why chemistry and biology are unaffected. All of this applies equally to other inertial frames of reference, which means all other star ships cruising at a constant velocity. This is all very reassuring and perhaps, as much as we could hope for. Nevertheless, deep in the detail, something is not quite right. The problem first appears in the theory of electromagnetism. We will examine this now, but we need not go into great detail.

The study of electric phenomena begins with electrostatics – the behaviour of stationary electric charges. From the work of Coulomb, Priestly and others it was established centuries ago that the force, f_e, between two static electric charges q_1 and q_2 separated by a distance r in a vacuum, is given by the equation (1):

$$f_e=\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}. \tag{1}$$

In this equation there appears a fundamental constant of nature ϵ_0 which is the so-called *electric constant*. It is, in essence, a measure of the permeability of empty space to the electric force and it can be determined aboard a star ship in much the same way as it is done on Earth. We can ignore the details, but the value obtained will be the same: 8.8542×10^{-12} farads per metre (in standard electromagnetic units).

The modern treatment of magnetism is centred on the magnetic effects of electric currents, rather than the physical magnets we are more familiar with as, for example, fridge ornaments. It derives from a similar experiment to that used to define the electrostatic force and is concerned with the magnetic force (per unit of length) f_m between two parallel conducting wires carrying a current I set a distance r apart in a vacuum. The force per unit length of the wire is given by Ampere's law:

$$f_m=\frac{\mu_0}{2\pi}\frac{I^2}{r}. \tag{2}$$

Once again we encounter a fundamental constant: μ_0, which in this case is the magnetic constant measuring the permeability of empty space to the magnetic force.
If we determined this by experiment on a star ship it will yield the same value as obtained on Earth: 1.2566×10^{-6} newtons per ampere squared (also in standard electromagnetic units).

Now, here is a remarkable thing: in his theory of electromagnetism, Maxwell showed that light is an electromagnetic phenomenon, and that the speed of light is given by a simple formula involving the two constants ϵ_0 and μ_0 which is:

$$c=\frac{1}{\sqrt{\epsilon_0\mu_0}}, \tag{3}$$

Where c is the velocity of light. A modern calculator will show that according to this formula, c has the value 2.9979×10^{8} metres per second, precisely what the velocity of light is known to be.

This result is a great triumph of theoretical thinking, but in it lurks a puzzle. The theory returns the value of the speed of light that is *universal* and without reference to the state of motion of any experimenter. So what is the velocity relative to? In Maxwell's day this was assumed to be a special medium called the *luminiferous ether* which filled all space. This is rather like the phenomenon of sound. Sound is the vibration of the air, so if there was no air, no sound would would be possible. Similarly, light was thought to be the vibration of the luminiferous ether. This seems a reasonable assumption, but there are problems if we dig deeper. With regard to sound, we know that air can move – we call this wind. It seems fair to ask if the ether can move? The answer seems to be no. When the air is moved by wind, sound propagation is affected, causing effects we can hear. If light was affected by 'ether winds' it would be obvious in our observations of distant stars and galaxies, but as we know, such effects are absent. It was therefore assumed that the luminiferous ether was fixed in space. Indeed, it became identified with the concept of Absolute Space, which Newton had postulated as the universal framework of his laws of motion.

But wait! If the ether is fixed in absolute space, it follows that for every star ship cruising through space, each one should find the speed of light differs according to its velocity relative to the ether. If this is the case, a star ship could measure the speed of light in its local frame and then determine its absolute speed though space, which is in violation of Galileo's Principle! Why should light be able to break this principle when no other physical phenomenon can? This strange state of affairs led scientists to think that something must be wrong with the theory of electromagnetism. The Dutch physicist Hendrick Lorentz sought to patch up the theory so that such strange results could be accommodated, but in this he was only partially successful. He made valuable progress, but in an *ad hoc* way. He had no physical explanation for what was going on and did not fully appreciate the profound implications.

It was Einstein who established the true, and revolutionary, explanation. Einstein thought it made better sense if the speed of light was absolutely universal - the same in all inertial reference frames. He therefore declared this to be a fundamental principle[4]. This would mean electromagnetism indeed obeyed Galileo's Principle, but at what price? His investigation revealed that, astonishingly, space and time cannot be what Newton (and everyone else) supposed and the idea that these were absolute and immutable would have to be abandoned to make progress.

A universal value for the speed of light, regardless of the motion of the observer, is indeed revolutionary. If you think otherwise, consider this situation. Two star ships are on a collision course, travelling in a straight line directly towards each other at a speed v. Star ship A flashes a beam of light at star ship B[5]. The light beam flies away from A at the speed c, and for this reason A expects that B, which is ploughing headlong into the light beam, will encounter the beam at a speed that equals c *plus* the speed v, with which B is travelling towards A. However, according to Einstein, B actually finds the beam has the velocity c. This applies even if $v=c$, which is very counter-intuitive, but absolutely true!

Einstein's theory is today called the "Special Theory of Relativity". It is *special* because it applies to inertial rather than *accelerated* frames[6]. Originally the theory appeared in a paper called: "On the electrodynamics of moving bodies," which shows that Einstein was addressing a problems with electromagnetism, which is what we have done here. He was not trying to revolutionise physics, he was merely trying to resolve a discrepancy between the theories of Newton and Maxwell.

We will now look in detail at how these ideas are handled using the mathematics of *transformations*.

3. The Galilean Transformation

The Galilean transformation describes the relationship between measurements of space and time in different inertial frames. It is the transformation that was used before the universality of the speed of light was established.

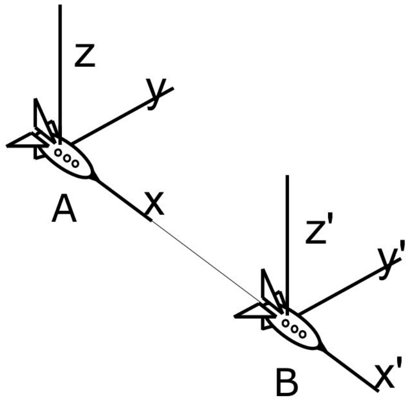
The basic assumption underpinning the description of the universe from the time of Newton was that time and space were absolute. Every observer in the universe, no matter what the frame of reference, had the same absolute measure of both. All inertial observers, no matter what their velocity, would say that the measured distance between two objects was the same and everyone would experience the passing of time at exactly the same rate. With this understanding, and confining ourselves to inertial frames, it is easy to arrive at the equations of transformation between different frames.

Figure 3. Co-linear Space Travel

4 Interestingly, the universality of the speed of light was experimentally verified by Michelson and Morley, who attempted to measure the speed of the Earth through the luminiferous ether and found it was not possible. The measured speed of light was evidently the same, no matter what velocity the moving Earth possessed. Surprisingly, Einstein did not know of this result, even though it was obtained before he developed his theory!

5 Presumably to warn them to get them out of the way!

6 In particular those arising from gravity.



To simplify matters, we will assume that we have two star ships A and B and they are travelling along the same straight line (as in Figure 3) with a constant relative velocity v, which simply means that ship B is faster than ship A by the amount v. (Note we do not assume ship A is stationary in any absolute sense, if indeed we knew how to specify this.) Each ship has its own frame of reference, which gives event coordinates (t,x,y,z) and (t',x',y',z') aboard ships A and B respectively, for an event that is observed by both of them. Note that the axes x and x' are co-linear, meaning that they lie along the same straight line. Also axes y and y' are parallel, as are axes z and z'. To complete the description, we specify that the times t and t' are both set to zero when when the origin of the coordinates of star ship A coincides with the origin of the coordinates of star ship B as the two ships pass by each other[7]. In other words, $t=t'=0$ when $(x,y,z)=(x',y',z')=(0,0,0)$. It can be shown that any arrangement of the star ships can be related to that shown in Figure 3 (see Appendix).

Given the arrangement of star ships in Figure 3, our quest is to find out how measurements of (t,x,y,z) on ship A relate to measurements of (t',x',y',z') on ship B, where both describe the same event. The result is known as the *Galilean transformation*, which is:

$$\text{(a)}\quad \begin{array}{l} t'=t \\ x'=x-vt \\ y'=y \\ z'=z \end{array}\quad , \qquad \text{(b)}\quad \begin{array}{l} t=t' \\ x=x'+vt' \\ y=y' \\ z=z' \end{array}\quad , \qquad (4)$$

in which the equations (4)(a) represent the transformation of coordinates (x,y,z) from the reference frame A to coordinates (x',y',z') in frame B and (4)(b) is the reverse transformation of the coordinates from frame B to frame A (the so called *inverse transformation*).

Where do these transformation equations come from? Well, given the assumed universality of time and space, they could hardly be anything else! The equivalence of the time coordinates t and t' follows directly from the universality of time. The

7 This arrangement actually implies the two ships collide(!) but we dismiss this as an unnecessary complication to what is only a simple mathematical abstraction. In thought experiments, nobody needs to get killed!

equivalence of y to y' and of z to z' follow from the universality of space and the construction given in Figure 3. The difference between x' and x in equations (a) follows from the fact that, from the reference frame of ship A, the origin of the x' coordinate is moving away from the origin of the x coordinate at the speed v, and so after time t it is displaced by the distance vt. The term $-vt$ simply corrects for this fact. A similar argument shows that, from the reference frame of star ship B, the origin of the x coordinate is moving away from that of the x' coordinate with a speed $-v$, so the correction in this case is $+vt'$.

This all seems perfectly sensible, so what is the problem? We can best show this if we consider how velocities change from one frame of reference to another.

4. The Galilean Transformation of Velocity

Because the two star ships in Figure 3 are in relative motion, when they both observe a passing comet (say), they will not determine it to be travelling with the same velocity. Therefore we would like to show that the velocity obtained in one reference frame is equivalent to the velocity in the other - in other words we need a *velocity transform*. For this purpose we must use the *differential* form (5) of the Galilean transformation (4) which is is easily obtained as

$$\text{(a)}\quad \begin{matrix} dt'=dt \\ dx'=dx-vdt \\ dy'=dy \\ dz'=dz \end{matrix} \quad , \qquad \text{(b)}\quad \begin{matrix} dt=dt' \\ dx=dx'+vdt' \\ dy=dy' \\ dz=dz' \end{matrix} \quad . \qquad (5)$$

Since $dt'=dt$, we can divide all the other equations by dt to obtain:

$$\text{(a)}\quad \begin{matrix} u_x'=u_x-v \\ u_y'=u_y \\ u_z'=u_z \end{matrix} \quad , \qquad \text{(b)}\quad \begin{matrix} u_x=u_x'+v \\ u_y=u_y' \\ u_z=u_z' \end{matrix} \quad . \qquad (6)$$

In these equations we have defined $\vec{u}=(u_x,u_y,u_z)$ as the *velocity vector* of the comet observed from the star ship A and $\vec{u}'=(u_x',u_y',u_z')$ as the comet's velocity vector observed from star ship B. Note we do not assume that the comet follows a co-linear trajectory as we did with the star ships, as it is an unnecessary simplification here.

Now, it is clear that in these equations that $u_y=u_y'$ and $u_z=u_z'$ but it is certainly *not* true that $u_x=u_x'$, which would strike us as being quite wrong. Indeed, *common sense* alone tells us that, because A and B move relative to each other, these two velocities cannot be equal. However, if instead of a comet, we were observing a beam of light, this transformation implies light would have a *different* speed in the different reference frames, so it cannot be right.

Why wasn't this discovered sooner? The answer partly lies in the observation that the equations (6) do give the right answers, usually to great precision, but in circumstances where the velocity v is small compared to light speed. But light travels at an enormous velocity, which puts it beyond normal experience. To correct for this problem we need to replace the Galilean transformation with something better: the Lorentz transformation.

5. Space-Time Diagrams

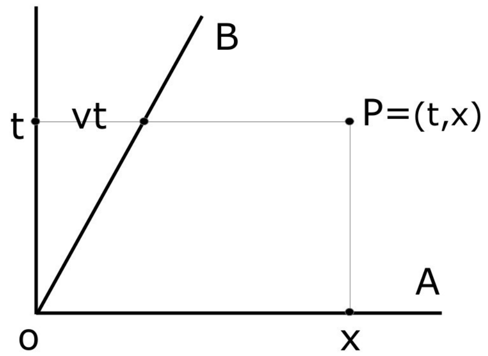


Figure 4. The Space-time Diagram

This is a good point to introduce the idea of a space-time diagram, which is a useful tool in discussing Einstein's theory. If we restrict ourselves to frames of reference that are in the co-linear configuration shown in Figure 3, then the diagrams are particularly simple but are still sufficient to lay bare the essential physics. To begin with, we can represent events observed from the frame of reference of star ship A as a simple two-dimensional Cartesian graph, with t as the vertical axis and x as the horizontal axis as shown in Figure 4. (We are thus ignoring what happens in the y and z directions for the time being.) This is a typical space-time diagram. An event happening at time t and at position x, is represented as a point P with coordinates (t,x), as you might expect. This is clearly a single, isolated event at a specific time and place.

But we can also show moving objects. For example, the line labelled B represents star ship B as it moves away from star ship A. Specifically, it shows how the origin of the coordinate system of B moves in relation to the origin of the coordinate system of A, given that both origins were coincident at $t=t'=0$. Since the velocity[8] v of B relative to A is a constant, the line is straight and points to the right – indicating that the spatial distance x' from the origin of B to the x-location of point P is *decreasing* with increasing time and indeed, it will eventually pass the point P. This behaviour is apparent in the equation for x' in (4)(a). This line is called the *world line* of B, since the star ship is stationary in its own frame of reference and its existence is confined to that line. Likewise the t axis is called the world-line of star ship A.

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8 Note we use the single number v to define the velocity of star ship B. We could of course properly use vector notation and write $\vec{v}$. However, because of the special alignment of the velocity in the co-linear configuration (Figure 3) the components of the vector are actually $(v,0,0)$, which contains no more information. So just using the number v is quite appropriate.

Figure 4 is thus a demonstration of the Galilean transformation, but as we have seen, this transformation does not correctly allow for the universal speed of light. So we must adapt this diagram to accommodate the behaviour of light. The first adaptation we must make is just a simple change of scale. This is necessary because the speed of light is so great. In conventional units (e.g. metres and seconds *etc.*) the world line of light would be indistinguishable from a horizontal line, being almost parallel to the t axis, which makes it hard to see clearly what's going on. Therefore, by convention, we use units of length and time that result in the velocity of light having the value unity i.e. $c=1$. This means world lines for light beams that are always at a slope of 45^o to the vertical on the space-time diagram. Such a choice of units we will refer to as *light-normalised units*, or just *light units* for short. Note that there is no physics in this choice; it is simply a matter of presentational convenience.

We can ensure the velocity of light is unity in two different ways. Firstly, we could define distance in terms of light-seconds, light-weeks or light years *etc.,* but retain the time unit as the second, week or year as appropriate. Thus the speed of light could be said to be one light-year per year for instance, which would give the speed of light as 1 in these units. We shall call this *time rendering,* since both space and time are in time units[9]. Alternatively we could define time in terms of distance by simply multiplying it by the velocity of light. Thus we could define a metre of time as the time it takes light to travel one metre, for example. Thus the speed of light would be one metre of distance per metre of time and hence $x, y,$ This we shall call *distance rendering,* since both space and time are in distance units. In both cases other speeds are scaled by the speed of light, so z becomes v/c or $v \rightarrow v/c$.

Either way, distance and time are expressed in the same units and as a consequence, velocities now have no units at all. Furthermore, since in both cases $c=1$ the parameter c no longer needs to make an explicit appearance in our formulae, which are therefore the same for both ways of rendering. It is not a problem that in one case formulae are rendered in terms of distance and in the other of time, but if we wish to convert formulae expressed in light units back into conventional units we can resort to some simple conversion rules:

- Replace the velocity variable by the term v/c i.e. $v \rightarrow v/c$.
- Then:
  For distance rendering (event components in distance units):
    - Replace the time variable t by the term ct i.e. $t \rightarrow ct$,
    - Replace the acceleration a by the term a/c^2 i.e. $a \rightarrow a/c^2$. (7)

  For time rendering (event components in time units):
    - Replace the distance variable x by x/c i.e.
    - Replace the acceleration a by the term a/c i.e. $a \rightarrow a/c$.
- The restored equations will be same for both distance and time rendering (after some rearranging).

9 Using light units means we can now define an event as a true four dimensional vector $\vec{e}=(t, x, y, z)$ since all components are now in the same units.

In what follows, we will mostly be using light units, since we are largely concerned with phenomena approaching the speed of light.

6. Simultaneity

In this section, to prepare for the Lorentz transformation, we explore Einstein's key discovery of the *relativity of simultaneity*, which states that events that coincide in time in one inertial frame (i.e. are *simultaneous*) do not necessarily coincide in time in another inertial frame. This proves to be a vital key in explaining why the Galilean transformation is inadequate. To see this we need to look at how light propagation appears in different inertial frames.

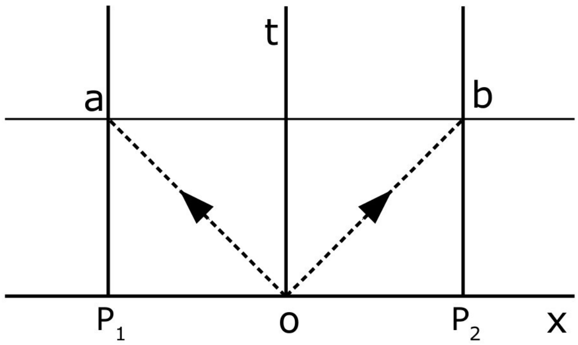

Figure 5: Light Propagation Aboard Star Ship A

On board star ship A, we set up the experiment shown in Figure 5. Two fixed points, P_1 and P_2, on the x-axis are set at equal distances either side of the origin at O. At a chosen time zero $(t=0)$, a flash of light is emitted from the origin, which propagates in both positive and negative directions of x to reach the points P_1 and P_2 at times a and b respectively. Since the light propagation is the same in the forwards and backwards directions, we will not be surprised to find that the light reaches the two points *simultaneously*, giving the result that $a=b$. The drawn line a – b in Figure 5 is thus a *line of simultaneity*, which is horizontal in the diagram and, most importantly, *parallel* to the x-axis. All events that occur anywhere along this line happen at the same time.

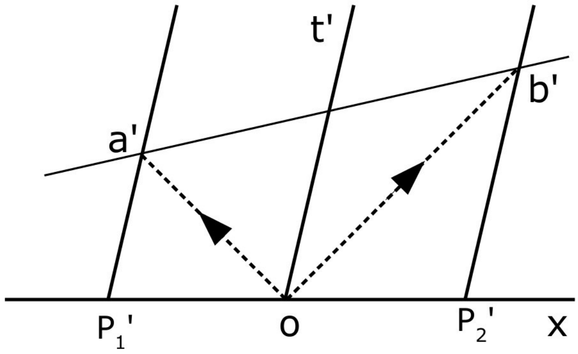

Figure 6: Light Propagation Aboard Star Ship B (as observed from A)

We now repeat the experiment aboard star ship B, but in this case we retain our view point aboard star ship A. The set-up is shown in Figure 6, in which star ship B is travelling *co-linearly* along the x-axis of star ship A (as in Figure 3). When the x' origin of B coincides with the x origin of A, there is a flash of light which propagates on to the points, P_1' and P_2', which are equidistant from the origin of x'. These points are stationary in frame B, but are moving in the frame of A and thus their world-lines slope to the right in Figure 6, parallel to the world-line of the x' origin of B.

Now, from the perspective of star ship B, the experiment is the same as that shown in Figure 5. The light must reach P_1' and P_2' simultaneously, since the relative velocity of ships A and B has no relevance to the propagation of light. But from the perspective of star ship A, this cannot be the case, as point P_1' is evidently advancing to meet the oncoming light and point P_2' is receding away from it. And so Figure 6 proves: from the reference frame of A, the events marked on Figure 6 as a' and b' do not occur at the same time. Nevertheless, we must also conclude that these events are indeed simultaneous in the reference frame of B. Thus line a' - b' is a line of simultaneity in reference frame B, but not in A.

This was Einstein's great discovery – simultaneity, and therefore time, is no longer absolute for all observers and different inertial frames (i.e. frames that are not stationary with respect to each other). Each must have a different measure of time. As we shall see, this also implies the measure of space must be different in different inertial frames. The Galilean transformation is thus no longer adequate and a new one is required. This leads us to the Lorentz transformation.

7. The Lorentz Transformation

Can we anticipate what a relativistically correct transformation is going to be like? Consider Figure 7, which is a new version of the space-time diagram of Figure 4. (Ignore, for now, the sloped dashed line O-P, which is used later.)

In Figure 7, the event P has the coordinates (t, x), in frame A, which are obtained by projections onto the axes O-t and O-x respectively. The O-t axis is the the world-line of star ship A, and the line P-t is a line of simultaneity drawn from P towards the t-axis, where it marks the time t. P-t is of course parallel to the x-axis. In the same way a line drawn from P to the x-axis, parallel to the t-axis marks the position x. In the reference frame of B it is the O-t' axis that represents the world-line of star ship B, so this must be the t'-axis, recording time in this frame. Also, we know that lines of simultaneity in frame B must be tilted with respect to the x-axis, so the x'-axis is a sloped line O-x'

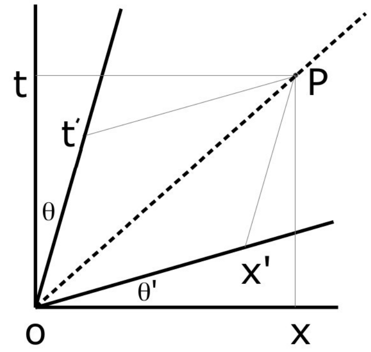

Figure 7. The Lorentz Transformation

To obtain the coordinates (t',x') in a manner resembling frame A, we must use projections from P to the axes O-t' and O-x' respectively, which are parallel to the axes of the frame B, as shown. From this we can well understand how the two different frames of reference can have different measures of time *and* space. Indeed it is important to note that the two coordinate systems do not necessarily have the same *scale*. This is quite a different picture from Figure 4, where we imposed the universality of space and time on both coordinate systems. The new transformation was first derived by Lorentz and is named after him, though it came from a different approach to Einstein's.

So we have some idea of how the Lorentz transformation has to work, but we do not yet have a *quantitative* transform. The first thing we need to know is precisely where to draw the x' axis on the diagram. We know it must pass through the origin $(0,0)$, but what is the angle θ' shown in Figure 7? Straight away we can say that θ' is the same as the angle θ. Why is this? Consider the case where event P lies on the world line of a light beam (indicated in Figure 7 by the dashed line). The world line of light must be at 45^o to the vertical, which means, it bisects the (right) angle between the t-axis and x-axis. The bisection means that anywhere along the light line any chosen time interval, Δt, equals its corresponding distance interval, Δx, such that $\Delta x/\Delta t=1$. The same must be true in the moving reference frame i.e. $\Delta x'/\Delta t'=1$. So it follows that the light line must bisect the angle between the axes t' and x' also. It then easily follows that $\theta'=\theta$.

As to the value of θ, we can see from Figure 7 that the angle is given by equation

$$tan\theta=vt/t=v, \quad (8)$$

where ^{r} is the velocity of star ship B with respect to A, in light units. One important insight we gain from the equality of the angles is that time and space coordinates are affected similarly by the Lorentz transformation if we work in light units.

Armed with this information we suggest the form for the Lorentz transformation as

$$\text{(a)}\quad \begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\quad , \qquad \text{(b)}\quad \begin{aligned} t &= \gamma(t' + vx') \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned}\quad , \tag{9}$$

in which γ is the *Lorentz factor,* a scaling constant we have yet to derive. Its introduction is necessary to allow for the possibility that the measurement of time and space in the frame B is different from frame A, as we have noted above. We must also use the same constant for the forward and reverse transforms, because different scaling constants would imply the two frames are not equivalent. Note that the transform equations for the x coordinates in both sets of formulae are similar to the Galilean transformation, except that we have allowed for the change of scale. The equations for t' and x' in (9)(a) have similar forms (as do their inverse transforms in (9)(b)), which reflects the observation above that these two coordinates are affected by the transformation in a similar way. We have also retained the equations for y' and z' from the Galilean transformation, since there is no experimental evidence that these coordinates are affected under the configuration of the co-linear travel shown in Figure 3.

The only remaining thing we need to know is the value of the constant γ. We can find this by exploiting the reversibility of the transformations. Since from (9)(a) we have

$$x' = \gamma(x - vt). \tag{10}$$

Then substituting for x and t from (9)(b) gives

$$\begin{aligned} x' &= \gamma(\gamma(x' + vt') - v\gamma(t' + vx')), \\ x' &= \gamma^2(x' + vt' - vt' - v^2 x'), \\ x' &= \gamma^2 x'(1 - v^2), \\ \gamma^2 &= 1/(1 - v^2), \end{aligned} \tag{11}$$

and hence

$$\gamma = 1/\sqrt{1 - v^2}. \tag{12}$$

So the full Lorentz transformation for the co-linear travel configuration is

$$\text{(a)}\quad \begin{aligned} t' &= \frac{1}{\sqrt{1-v^2}}(t - vx) \\ x' &= \frac{1}{\sqrt{1-v^2}}(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \quad , \qquad \text{(b)}\quad \begin{aligned} t &= \frac{1}{\sqrt{1-v^2}}(t' + vx') \\ x &= \frac{1}{\sqrt{1-v^2}}(x' + vt') \\ y &= y' \\ z &= z' \end{aligned} \quad . \qquad (13)$$

This is not the form normally shown in text books because we are using light units. To convert the formulae to conventional units, we use the conversion rules (7) given above i.e. the substitutions $t \to ct$ and $v \to v/c$. With a little rearranging the equations (13) take the form

$$\text{(a)}\quad \begin{aligned} t' &= \gamma\left(t - v\,x/c^2\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \quad , \qquad \text{(b)}\quad \begin{aligned} t &= \gamma\left(t' + vx'/c^2\right) \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned} \quad . \qquad (14)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

The formulae (14) help to explain why the Galilean transformation was thought correct for so long. If we assume $v \ll c$ then v/c is a negligibly small number in most practical cases and the formulae can be shown to approximate the Galilean transformation (4) very accurately.

8. Relativistic Effects of the Lorentz Transformations

According to the Lorentz transformation we should not expect star ships A and B, which move with a constant velocity relative to each other, to measure time and space in the same way. Here we give a quantitative measure of how different they actually are.

With regard to the measurement of length consider the experiment shown in Figure 8.

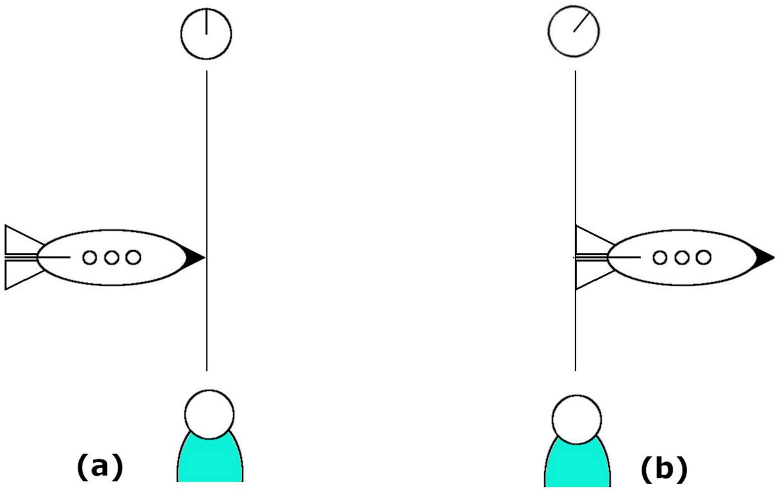

Figure 8: Measuring the Length of Space Ship B

The objective in Figure 8 is to measure the length of star ship B, travelling at the velocity v, as it passes a fixed point in the reference frame of star ship A. An observer in frame A notes the time, t_1, when the nose of star ship B reaches a fixed line at a position x_1 (say) - as in diagram (a), then waits until the tail of star ship B reaches the same fixed line - as in diagram (b), and notes the time, t_2, when this occurs. From the time difference, $t=t_2-t_1$, and the velocity, v, the length of star ship B is determined as

$$L=vt. \tag{15}$$

This is simple enough, but let us look at the experiment from the frame of reference B. Firstly we note that in frame A at the time t_2, the nose of star ship B is at the position $x_2=L+x_1$ and the tail is at x_1. This gives us the space-time events $(t_{2,}x_1)$ and $(t_{2,}x_2)$ which define the two ends of the star ship simultaneously in the frame of A.

Now, using the Lorentz transformation (13) we have

$$\begin{aligned} x_1'&=\gamma(x_1-vt_2) \\ x_2'&=\gamma(x_2-vt_2) \end{aligned}, \tag{16}$$

and so

$$x_2'-x_1'=\gamma(x_2-x_1-v(t_2-t_2)). \tag{17}$$

If we define $L=x_2-x_1$ and $L'=x_2'-x_1'$ i.e. the distance between the nose and tail of the star ship in the frames A and B, it follows that

$$L'=\gamma L. \tag{18}$$

Clearly, the two lengths, L' and L, differ when $v>0$ and indeed $L'>L$ in all such cases. Since L' is the true or *proper* length of star ship B, measured in its own frame of reference, we can see that, when it moves through the frame of reference A, it is observed to have a length shorter than its proper length. This phenomenon is known as the Lorentz-Fitzgerald contraction. It is encapsulated in the statement: *moving objects appear shorter.* A more common form of equation (18) is

$$L=\gamma^{-1}L_0, \tag{19}$$

where L is the observed length of a moving object, L_0 is its proper (or resting) length and $\gamma^{-1}=\sqrt{(1-v^2)}$. Incidentally, it is worth noting that the length contraction only occurs in the the direction of motion; the perpendicular y and z directions are unaffected.

There is something else we can learn from this experiment. We note that at time t_1 in frame A the nose of star ship B is at location x_1, and that at time t_2 it is at location $x_2=x_1+vt$, where $t=t_2-t_{1.}$ We ask how the events $(t_{1,}x_1)$ and $(t_{2,}x_2)$ in the frame of A transform to the frame of B. First we look at the x-coordinates; the Lorentz transformation gives

$$\begin{aligned} x_1' &= \gamma(x_1-vt_1) \\ x_2' &= \gamma(x_2-vt_2) \end{aligned} . \tag{20}$$

(Note that x_1' is not the same as in (16), as the corresponding time coordinate is different i.e. this is a different event in space-time.) From (20) we get

$$\begin{aligned} x_2'-x_1' &= \gamma(x_2-x_1-v(t_2-t_1)) \\ x_2'-x_1' &= \gamma(L-vt) \end{aligned} , \tag{21}$$

and since $L=vt$ in the frame of A, this means that

$$x_2'-x_1'=0 \quad \text{or} \quad x_2'=x_1'. \tag{22}$$

So the x' coordinate is stationary in the frame of B, which is of course consistent with it being the fixed location of the nose of the star ship in frame B.

Now looking at the time coordinates in this case, the Lorentz transformation gives

$$\begin{aligned} t_1' &= \gamma(t_1-v\,x_1) \\ t_2' &= \gamma(t_2-v\,x_2) \end{aligned} . \tag{23}$$

Subtracting one equation from the other we obtain

$$\begin{aligned} t_2'-t_1' &= \gamma(t_2-t_1-v(x_2-x_1)) \\ t' &= \gamma(t-vL) \end{aligned} \tag{24}$$

and since $L=vt$, this gives

$$t' = \gamma t\left(1 - v^2\right)$$
$$t' = \gamma^{-1} t \quad . \qquad (25)$$

This result shows that when $v > 0, t' < t,$ i.e. the time interval t in frame A, is larger than the corresponding time interval t' in the frame B. In other words time in the moving frame falls behind that in the stationary frame. This phenomenon is known as *time dilation.* The equation (25) is often written as

$$t_0 = \gamma^{-1} t, \qquad (26)$$

in which t_0 represents the *proper time*, i.e. the time measured by a clock in the moving frame, while t is the corresponding time measured by a clock in the stationary frame. The relationship (26) is encapsulated in the statement: *moving clocks run slow.*

9. The Relativistic Doppler Effect

Since the speed of light is constant for all star ships, we may wonder what effect this has on the Doppler shift, which is an invaluable tool in astronomy for determining the speed with which objects approach or recede from an observer. In the case of space travel, its importance lies in the determination of the speed of a star ship's approach to its destination.

Suppose star ship A is stationary with respect to a distant star and receives an electromagnetic wave from the star, which is lying along the positive x-direction. This wave has a frequency f and wavelength λ and we note that for all wave motion the following relation holds

$$f\lambda = c \quad \text{or} \quad f\lambda = 1 \ \text{(in light units).} \qquad (27)$$

Star ship B travelling along the x-direction with velocity v receives the same signal, but it records a different wavelength λ' and frequency f', though again we have $f'\lambda' = 1$ (in light units). The different frequency and wavelength is due to Doppler shifting, and we need to account for this using relativity theory.

For simplicity is is adequate to assume the electromagnetic wave observed by A takes the form

$$\Psi = A \sin\left(2\pi\left[\frac{x}{\lambda} + ft\right]\right), \qquad (28)$$

where λ and f are the wavelength and frequency respectively and A is an amplitude constant. To obtain the corresponding form for star ship B, we must transform variables x and t using the Lorentz transformations (13). This gives

$$\Psi = A\sin\left(2\pi\gamma\left[\frac{(x'+vt')}{\lambda}+f(t'+vx')\right]\right). \tag{29}$$

Gathering together terms in x' and t' and simplifying the result using $f'\lambda'=1$, leads to

$$\Psi = A\sin\left(2\pi\gamma(1+v)\left[\frac{x'}{\lambda}+ft'\right]\right), \tag{30}$$

which, by comparison with (28) implies that

$$\lambda'=\frac{\lambda}{\gamma(1+v)} \quad \text{and} \quad f'=f\gamma(1+v). \tag{31}$$

We can write $\gamma(1+v)$ in the form:

$$\gamma(1+v)=\frac{(1+v)}{\sqrt{(1-v^2)}}=\sqrt{\frac{1+v}{1-v}}, \tag{32}$$

so equations (31) become

$$\lambda'=\lambda\sqrt{\frac{1-v}{1+v}} \quad \text{and} \quad f'=f\sqrt{\frac{1+v}{1-v}}. \tag{33}$$

From which we see that, when B is travelling *towards* the stellar emission (i.e. v is positive), the frequency f' is *increased* and the wavelength *decreased*, when compared with A, so it is *blue-shifted* in the common terminology. Conversely, when B is travelling *away from* the stellar emission (i.e. v is negative), the frequency f' is *decreased* and the wavelength *increased*, with respect to A, which means it is *red-shifted*.

In conventional units the formulae (33) are written as

$$\lambda'=\lambda\frac{\sqrt{(c-v)}}{\sqrt{(c+v)}} \quad \text{and} \quad f'=f\frac{\sqrt{(c+v)}}{\sqrt{(c-v)}}. \tag{34}$$

Also when v is small (i.e. $v \ll c$ in conventional units) we can easily simplify these formulae to give their classical forms. Equations (34) become

$$\delta\lambda=-\lambda v \quad \text{and} \quad \delta f=fv, \tag{35}$$

where $\delta\lambda$ and δf are the *shifts* in wavelength and frequency respectively on changing reference frames from A to B i.e.

$$\delta\lambda=\lambda'-\lambda \quad \text{and} \quad \delta f=f'-f. \tag{36}$$

In conventional units equations (35) are

$$\delta\lambda = -\lambda \frac{v}{c} \quad \text{and} \quad \delta f = f \frac{v}{c}. \tag{37}$$

Hence, armed with knowledge of the spectroscopy of the star, a star ship can work out its speed of approach towards it.

10. The Space-Time Interval

The space time interval is a simple property to calculate, but it offers profound insight into how relativity works, as this section shows.

In ordinary Cartesian spatial coordinates, we can calculate the separation, Δr, between two points, (x_1, y_1, z_1) and (x_2, y_2, z_2), using Pythagoras theorem in three dimensions:

$$\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2, \tag{38}$$

where

$$\begin{aligned} \Delta x &= x_2 - x_1 \\ \Delta y &= y_2 - y_1 \\ \Delta z &= z_2 - z_1 \end{aligned} \quad . \tag{39}$$

This is unsurprising, but what is interesting about this result is that the value of Δr^2 obtained is independent of which Cartesian frame of reference we use to define the coordinates[10]. (There is, in principle, an infinite number of them, differing in the location of the origin and the orientation in space.) It is thus an invariant property of the Euclidean space in which it is calculated. The question arises as to what, if anything, is an invariant property of space-time?

It turns out that in space-time the corresponding equation to (38) is

$$\Delta \tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2, \tag{40}$$

in which the term Δt, is given by analogy with (39) as:

$$\Delta t = t_2 - t_1. \tag{41}$$

The parameter $\Delta \tau$, is called the *space-time interval,* and it is the separation of two events in the four dimensions of space-time. That $\Delta \tau^2$ in equation (40) is an invariant is easily demonstrated. Using the Lorentz transformation (13) we can write (40) as

10 Provided the coordinates are not scaled differently in different frames.

$$\begin{aligned}\Delta\tau^2 &= \gamma^2\left((\Delta t'+v\Delta x')^2-(\Delta x'+v\Delta t')^2\right)-\Delta y'^2-\Delta z'^2\\ \Delta\tau^2 &= \gamma^2\left(\Delta t'^2-v^2\Delta t'^2-\Delta x'^2+v^2\Delta x'^2\right)-\Delta y'^2-\Delta z'^2\\ \Delta\tau^2 &= \gamma^2\left(\Delta t'^2(1-v^2)-\Delta x'^2(1-v^2)\right)-\Delta y'^2-\Delta z'^2\\ \Delta\tau^2 &= \Delta t'^2-\Delta x'^2-\Delta y'^2-\Delta z'^2\end{aligned} \quad . \qquad (42)$$

So the invariance of $\Delta\tau^2$ is established. For brevity we may write equation (40) in the alternative form

$$\Delta\tau^2=\Delta t^2-\Delta r^2, \qquad (43)$$

where Δr^2 is given by (38). In common physical units, as opposed to light units, it is written as

$$c^2\Delta\tau^2=c^2\Delta t^2-\Delta r^2, \quad \text{or} \quad \Delta\tau^2=\Delta t^2-\Delta r^2/c^2. \qquad (44)$$

An interesting property of equation (40) is that it makes no mention of the velocity of the frame of reference - either its magnitude or its direction (even the important parameter γ is absent!), and there is no apparent distinction made between the spatial coordinates (x, y, z). Its use is thus not confined to the co-linear configuration we have been using so far. In fact it applies to *all* inertial frames regardless of their relative motion, location or orientation. This makes the formula particularly powerful for investigating the general properties of inertial frames.

For example, imagine a star ship travelling at a constant velocity through our (inertial) frame of reference. If its relative velocity is v, then the distance Δr that it travels in a time interval Δt is

$$\Delta r=v\Delta t. \qquad (45)$$

Thus we can write the space-time interval as

$$\Delta\tau^2=\Delta t^2-v^2\Delta t^2. \qquad (46)$$

In the reference frame of the star ship itself however, there is no motion i.e. $v=0$, so we have

$$\Delta\tau^2=\Delta t'^2, \qquad (47)$$

which incidentally shows that $\Delta\tau$ is equivalent to the the *proper time* in the moving frame of reference. Combining (46) and (47) gives

$$\Delta t'^2=\Delta t^2-v^2\Delta t^2, \quad \text{or} \quad \Delta t'=\Delta t\sqrt{1-v^2}, \qquad (48)$$

which is the time dilation formula (26) once again.

Alternatively, in our frame, we can lay two markers a distance $\Delta r=L$ apart along the trajectory of the star ship, which crosses this length in the time $\Delta t=L/v$. The squared space-time interval of crossing is thus

$$\Delta\tau^2 = L^2/v^2 - L^2, \tag{49}$$

Aboard the star ship frame the markers are measured as L' apart and the distance is passed by in the time $\Delta t' = L'/v$. However both markers are observed to pass by from the same position of the local frame, so $\Delta r' = 0$. The squared space-time interval of passing is thus given as

$$\Delta\tau^2 = L'^2/v^2. \tag{50}$$

Multiplying both (49) and (50) by v^2 and combining gives

$$L'^2 = L^2(1 - v^2) \quad \text{or} \quad L' = L\sqrt{1 - v^2}. \tag{51}$$

This is the Lorentz-Fitzgerald contraction of the distance markers observed in the star ship frame. Results (48) and (51) confirm the equivalence of (40) to the Lorentz transformation.

Further understanding of the space-time interval can be obtained by considering three cases of interval:

1. $\Delta\tau^2 = 0$. This occurs whenever $\Delta t = |\Delta r|$, which means $|\Delta r|/\Delta t = 1$, or the speed of light. The interval is thus described as *light-like*. The events at either end of the interval are at the limit of *causality* – they can only be causally related to each other if whatever propagates between them travels at light speed, which implies it is *non-material*.

2. $\Delta\tau^2 > 0$. So $\Delta t > |\Delta r|$, and $|\Delta r|/\Delta t < 1$, which is less than the speed of light. The interval is then said to be *time-like*. Events marking the ends of the interval can be causally related, since a moving object can connect both events without violating the speed of light. In this case $\Delta\tau$ represents the *proper time* of the moving object.

3. $\Delta\tau^2 < 0$. So $\Delta t < |\Delta r|$, and $|\Delta r|/\Delta t > 1$, which exceeds the speed of light. The interval is then said to be *space-like*. Events marking the ends of the interval cannot be causally related, since nothing can connect them without violating the speed of light. In this case $\sqrt{-\Delta\tau^2}$ represents the *proper distance* between the two events i.e. the spatial distance between the events in a reference frame where they are simultaneous and stationary.

These cases represent extremely important properties of space-time.

11. The Lorentz Transformation of Velocity

To work out how velocity is affected by the Lorentzian Transformation, again imagining that we have measured a comet's velocity with respect to the different star ships A and B. We first write the transformation equations (13) in differential form:

$$\text{(a)}\quad \begin{aligned} dt' &= \gamma(dt - v\,dx) \\ dx' &= \gamma(dx - v\,dt) \\ dy' &= dy \\ dz' &= dz \end{aligned} \quad , \qquad \text{(b)}\quad \begin{aligned} dt &= \gamma(dt' + v\,dx') \\ dx &= \gamma(dx' + v\,dt') \\ dy &= dy' \\ dz &= dz' \end{aligned} \quad . \tag{52}$$

Now, dividing the space differentials by the appropriate time differential in each case gives

$$\text{(a)}\quad \begin{aligned} \frac{dx'}{dt'} &= \frac{(dx - v\,dt)}{(dt - v\,dx)} \\ \frac{dy'}{dt'} &= \frac{dy}{\gamma(dt - v\,dx)} \\ \frac{dz'}{dt'} &= \frac{dz}{\gamma(dt - v\,dx)} \end{aligned} \quad , \qquad \text{(b)}\quad \begin{aligned} \frac{dx}{dt} &= \frac{(dx' + v\,dt')}{(dt' + v\,dx')} \\ \frac{dy}{dt} &= \frac{dy'}{\gamma(dt' + v\,dx')} \\ \frac{dz}{dt} &= \frac{dz'}{\gamma(dt' + v\,dx')} \end{aligned} \quad . \tag{53}$$

Taking dt or dt' outside the brackets in each case and replacing the time derivatives of distances with the corresponding velocities leads to

$$\text{(a)}\quad \begin{aligned} u_x' &= \frac{(u_x - v)}{(1 - v\,u_x)} \\ u_y' &= \frac{u_y}{\gamma(1 - v\,u_x)} \\ u_z' &= \frac{u_z}{\gamma(1 - v\,u_x)} \end{aligned} \quad , \qquad \text{(b)}\quad \begin{aligned} u_x &= \frac{(u_x' + v)}{(1 + v\,u_x')} \\ u_y &= \frac{u_y'}{\gamma(1 + v\,u_x')} \\ u_z &= \frac{u_z'}{\gamma(1 + v\,u_x')} \end{aligned} \quad , \tag{54}$$

where $\vec{u} = (u_x, u_y, u_z)$ is the comet's velocity vector observed from frame A and $\vec{u}' = (u_x', u_y', u_z')$ is its velocity vector observed from frame B.

If we suppose that $u_x = 1$ and $u_y = u_z = 0$ i.e. the comet travels with the speed of light in the x-direction. Then according to (54)(a)

$$u_x' = (1 - v)/(1 - v) = 1 \quad \text{and} \quad u_y' = u_z' = 0 \tag{55}$$

Thus anything travelling with the speed of light in frame A also travels with the speed of light in frame B! This corrects the problem we saw with the Galilean transformation and ensures that the velocity of light is universal.

Equations (54) can be converted to normal units using the rules (7) to give

$$u_x' = \frac{(u_x - v)}{(1 - v u_x/c^2)} \qquad u_x = \frac{(u_x' + v)}{(1 + v u_x'/c^2)}$$

$$u_y' = \frac{u_y}{\gamma(1 - v u_x/c^2)} \quad \text{and} \quad u_y = \frac{u_y'}{\gamma(1 + v u_x'/c^2)} \tag{56}$$

$$u_z' = \frac{u_z}{\gamma(1 - v u_x/c^2)} \qquad u_z = \frac{u_z'}{\gamma(1 + v u_x'/c^2)}$$

12. The Lorentz Transformation of Acceleration

To obtain the Lorentz transform for acceleration, we proceed in much the same way as we did for velocity, with the differential form of the Lorentz time transform (13) and the differential forms of the velocity transform (54):

$$\begin{aligned} dt' &= \gamma(dt - vdx) \\ du_x' &= \frac{du_x}{(1 - vu_x)} + \frac{(u_x - v)}{(1 - vu_x)^2} vdu_x \\ du_y' &= \frac{du_y}{\gamma(1 - vu_x)} + \frac{u_y}{\gamma(1 - vu_x)^2} vdu_x \\ du_z' &= \frac{du_z}{\gamma(1 - vu_x)} + \frac{u_z}{\gamma(1 - vu_x)^2} vdu_x \end{aligned} \quad \& \quad \begin{aligned} dt &= \gamma(dt' + vdx') \\ du_x &= \frac{du_x'}{(1 + vu_x')} - \frac{(u_x' + v)}{(1 + vu_x')^2} vdu_x' \\ du_y &= \frac{du_y'}{\gamma(1 + vu_x')} - \frac{u_y'}{\gamma(1 + vu_x')^2} vdu_x' \\ du_z &= \frac{du_z'}{\gamma(1 + vu_x')} - \frac{u_z'}{\gamma(1 + vu_x')^2} vdu_x' \end{aligned} . \tag{57}$$

(a) (b)

Dividing the differential velocities in (57) by the differential times leads to

$$\begin{aligned} \dot{u}_x' &= \frac{\dot{u}_x}{\gamma(1 - vu_x)^2} + \frac{(u_x - v)}{\gamma(1 - vu_x)^3} v\dot{u}_x \\ \dot{u}_y' &= \frac{\dot{u}_y}{\gamma^2(1 - vu_x)^2} + \frac{u_y}{\gamma^2(1 - vu_x)^3} v\dot{u}_x, \\ \dot{u}_z' &= \frac{\dot{u}_z}{\gamma^2(1 - vu_x)^2} + \frac{u_z}{\gamma^2(1 - vu_x)^3} v\dot{u}_x \end{aligned} \quad \& \quad \begin{aligned} \dot{u}_x &= \frac{\dot{u}_x'}{\gamma(1 + vu_x')^2} - \frac{(u_x' + v)}{\gamma(1 + vu_x')^3} v\dot{u}_x' \\ \dot{u}_y &= \frac{\dot{u}_y'}{\gamma^2(1 + vu_x')^2} - \frac{u_y'}{\gamma^2(1 + vu_x')^3} v\dot{u}_x', \\ \dot{u}_z &= \frac{\dot{u}_z'}{\gamma^2(1 + vu_x')^2} - \frac{u_z'}{\gamma^2(1 + vu_x')^3} v\dot{u}_x' \end{aligned} \tag{58}$$

(a) (b)

These equations are (a) the transformation of the acceleration vector $\vec{\dot{u}} = (\dot{u}_x, \dot{u}_y, \dot{u}_y)$ of an object observed in the inertial frame of a star ship A into the acceleration vector $\vec{\dot{u}}' = (\dot{u}_x', \dot{u}_y', \dot{u}_y')$ of the object observed in the inertial frame of a star ship B, moving with a constant velocity v relative to A and; (b) the corresponding inverse transformation. Note that in equations (58) neither star ship A nor B is undergoing acceleration, only the object being observed. For reference, according to conversion scheme (7) the equations (58) are written in normal units as

$$\dot{u}_x' = \frac{\dot{u}_x}{\gamma\left(1 - vu_x/c^2\right)^2} + \frac{\left(u_x - v\right)}{\gamma\left(1 - vu_x/c^2\right)^3}\frac{v}{c^2}\dot{u}_x x$$
$$\dot{u}_y' = \frac{\dot{u}_y}{\gamma^2\left(1 - vu_x/c^2\right)^2} + \frac{u_y}{\gamma^2\left(1 - vu_x/c^2\right)^3}\frac{v}{c^2}\dot{u}_x, \quad \text{(a)}$$
$$\dot{u}_z' = \frac{\dot{u}_z}{\gamma^2\left(1 - vu_x/c^2\right)^2} + \frac{u_z}{\gamma^2\left(1 - vu_x/c^2\right)^3}\frac{v}{c^2}\dot{u}_x$$

(59)

$$\dot{u}_x = \frac{\dot{u}_x'}{\gamma\left(1 + vu_x'/c^2\right)^2} - \frac{\left(u_x' + v\right)}{\gamma\left(1 + vu_x'/c^2\right)^3}\frac{v}{c^2}\dot{u}_x' x$$
$$\dot{u}_y = \frac{\dot{u}_y'}{\gamma^2\left(1 + vu_x'/c^2\right)^2} - \frac{u_y'}{\gamma^2\left(1 + vu_x'/c^2\right)^3}\frac{v}{c^2}\dot{u}_x', \quad \text{(b)}$$
$$\dot{u}_z = \frac{\dot{u}_z'}{\gamma^2\left(1 + vu_x'/c^2\right)^2} - \frac{u_z'}{\gamma^2\left(1 + vu_x'/c^2\right)^3}\frac{v}{c^2}\dot{u}_x'$$

Now, suppose that the object undergoing acceleration is the star ship B. Can the equations above be adapted to this circumstance? The answer is yes, provided we recognise that the *instantaneous* acceleration of an object that is stationary in the frame of star ship B is the same as the *instantaneous* acceleration of star ship B itself. So the two can be described by the same equations. We are in fact proposing to measure the so called *proper acceleration,* which is what is experienced on board star ship B. Note that a star ship measuring its own acceleration does not violate Galileo's principle because acceleration generally results from a physical force (Newton's second law), which, unlike velocity, is detectable[11].

We now consider the case of an instantaneous acceleration vector $\dot{\vec{u}}' = \left(\dot{u}_x', \dot{u}_y', \dot{u}_y'\right)$ determined by star ship B and show how this is observed from the star ship A which remains cruising at constant velocity. We first note that a stationary object in frame B has a velocity $u_x' = u_y' = u_z' = 0$ so putting this information into (58)(b) leads to

$$\dot{u}_x = \gamma^{-3}\dot{u}_x'$$
$$\dot{u}_y = \gamma^{-2}\dot{u}_y', \qquad (60)$$
$$\dot{u}_z = \gamma^{-2}\dot{u}_z'$$

We note at once that the acceleration $\dot{\vec{u}} = \left(\dot{u}_x, \dot{u}_y, \dot{u}_z\right)$, is not the same as the acceleration $\dot{\vec{u}}' = \left(\dot{u}_x', \dot{u}_y', \dot{u}_z'\right)$, which differs in both magnitude and direction. This is a strange result in the light Newtonian physics, where the acceleration is the same in both reference frames. This clearly requires a rethink of Newton's laws of motion, which we discuss later. Before that however, we need to discuss the property of momentum.

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11 Gravity is the one exception to this, since it turns out to be a a property of space-time itself.

13. Relativistic Momentum

Momentum is a key concept in Newtonian physics. It is defined as the product of a body's mass, m, with its velocity (u_x, u_y, u_z) as in:

$$\vec{p}=m\vec{u} \tag{61}$$

in which $\vec{p}=(p_x, p_y, p_z)$ is the so-called momentum vector, $\vec{u}=(u_x, u_y, u_z)$ is the velocity vector and the mass m is a fixed (i.e. constant) quantity. Now, according to Newton's laws of motion any force, however weak, applied continuously to a free body in space will eventually push it beyond the speed of light, which is forbidden by relativity theory. Nevertheless, Newton's laws are extremely accurate when the velocities concerned are much less than the speed of light, which suggests that the laws can be modified to allow for the effects of relativity at high speed. Doing this requires a closer examination of the momentum.

We start by considering the star ships A and B cruising at constant relative velocity v through space, in the co-linear configuration shown in Figure 3. We have already established the Lorentz transformation (13) that resolves their different views of space and time and introduced the important Lorentz factor, γ, which is given by equation (12).

Into this set-up we now introduce a comet travelling with velocity $\vec{u}=(u_x, u_y, u_z)$ with respect to star ship A, and velocity $\vec{u}'=(u_x', u_y', u_z')$ with respect to star ship B. This velocity need not be co-linear with the travel of the star ships, but is assumed constant. The comet is itself an inertial reference frame, so we can define two additional Lorentz factors, γ_u and γ_u', as in

$$\gamma_u=\left(1-u_x^2-u_y^2-u_z^2\right)^{-1/2}, \tag{62}$$

where γ_u is the Lorentz factor relating the comet to star ship A and

$$\gamma_{u'}=\left(1-u_x^2{}'-u_y^2{}'-u_z^2{}'\right)^{-1/2}, \tag{63}$$

where $\gamma_{u'}$ is the Lorentz factor relating the comet to star ship B.

We can now define two velocity dependent masses, m and m', as in

$$m=\gamma_u m_o \quad \text{and} \quad m'=\gamma_{u'} m_o, \tag{64}$$

where m_o is a mass constant (independent of velocity), the meaning of which we will assign later.

Masses m and m' can be used to define momentum components

$$\vec{p}=m\vec{u} \quad \text{and} \quad \vec{p}'=m'\vec{u}' \tag{65}$$

which, we propose, represent the *relativistic* momentum vector of the comet as determined by star ships A and B respectively. We need now to prove this.

Via the velocity transformation (54)(a), the velocity $\vec{u}'=(u_x', u_y', u_z')$ may be transformed into the velocity $\vec{u}=(u_x, u_y, u_z)$ and thus equation (63) can be written as

$$\gamma_{u'}=\left(1-\frac{(u_x-v)^2}{(1-u_x v)^2}-\frac{u_y^2}{\gamma^2(1-u_x v)^2}-\frac{u_z^2}{\gamma^2(1-u_x v)^2}\right)^{-1/2}, \tag{66}$$

where γ is defined in equation (12). After taking the term $1/\gamma^2(1-u_x v)^2$ outside the square root, some further manipulation gives

$$\gamma_{u'}=\gamma(1-u_x v)(1-u_x^2-u_y^2-u_z^2)^{-1/2}, \tag{67}$$

which we see from (62) is

$$\gamma_{u'}=\gamma\gamma_u(1-u_x v). \tag{68}$$

Using this result we can write the velocity transformation equations (54)(a) in a new way. Multiplying both sides of the equations (54)(a) by γ_u' results in

$$\begin{aligned} \gamma_{u'}u_x' &= \gamma\gamma_u(u_x-v) \\ \gamma_{u'}u_y' &= \gamma_u u_y \\ \gamma_{u'}u_z' &= \gamma_u u_z \end{aligned} \quad . \tag{69}$$

We now use the equations (64), (65), (68) and (69) to derive the following:

$$\begin{aligned} m' &= m_o\gamma_{u'}=m_o\gamma\gamma_u(1-u_x v)=\gamma m(1-u_x v)=\gamma(m-p_x v) \\ p'_x &= m_o\gamma_{u'}u_x'=m_o\gamma\gamma_u(u_x-v)=\gamma m(u_x-v)=\gamma(p_x-mv) \\ p'_y &= m_o\gamma_{u'}u_y'=m_o\gamma_u u_y=mu_y=p_y \\ p'_z &= m_o\gamma_{u'}u_z'=m_o\gamma_u u_z=mu_z=p_z \end{aligned} \quad . \tag{70}$$

We cannot help but notice that equations (70) are the same as the Lorentz transformation (13)(a), except that the time has been replaced by mass and positions replaced by momenta. (We can also obtain the inverse transformation resembling (13)(b), in much the same way.) Exploiting this isomorphism we write the mass-momentum Lorentz transformation as

$$\text{(a)} \quad \begin{aligned} m' &= \gamma(m-v\,p_x) \\ p_x' &= \gamma(p_x-vm) \\ p_y' &= p_y \\ p_z' &= p_z \end{aligned} \quad , \qquad \text{(b)} \quad \begin{aligned} m &= \gamma(m'+v\,p_x') \\ p_x &= \gamma(p_x'+vm') \\ p_y &= p_y' \\ p_z &= p_z' \end{aligned} \quad . \tag{71}$$

These equations show that the masses m and m' and the momenta (p_x, p_y, p_z) and

(p_x', p_y', p_z') transform consistently under the Lorentz transformation, but can we properly identify them as the relativistic counterparts of the classical mass and momentum?

If we examine the second equation of (71)(a) we see that for small v (i.e. $\gamma \approx 1$) the equation reduces to

$$p_x' = p_x - mv, \tag{72}$$

which shows that, *in the classical limit*, the momentum p_x' in the moving frame B is composed of the momentum p_x in the stationary frame A, minus the momentum the mass m acquires from being in a moving frame with velocity v. This is wholly consistent with classical mechanics. Furthermore, the presence of the Lorentz factor in the relativistic equation, together with the corresponding inverse equation in (71)(b) (which also behaves correctly in the classical limit) shows that the momentum and mass are behaving consistently in a proper relativistic manner. On this basis we should accept the proposed interpretation with confidence.

Finally we note once again that the equations (71) are appropriate only for light units. To set them in more conventional units we use the conversion rules (7) and exploit the isomorphism with equations (13). With a little rearrangement we obtain

$$\text{(a)} \quad \begin{array}{l} m' = \gamma\left(m - v\,p_x/c^2\right) \\ p_x' = \gamma\left(p_x - vm\right) \\ p_y' = p_y \\ p_z' = p_z \end{array} \quad , \qquad \text{(b)} \quad \begin{array}{l} m = \gamma\left(m' + v\,p_x'/c^2\right) \\ p_x = \gamma\left(p_x' + vm'\right) \\ p_y = p_y' \\ p_z = p_z' \end{array} \quad , \tag{73}$$

where γ is given by equation (12).

For completeness we need to account for the mass parameter m_o introduced in equation (65). What is its meaning? From the definition (64) we see that as $(u_x, u_y, u_z) \rightarrow (0,0,0)$ then $\gamma_u \rightarrow 1$. In which case, the comet is stationary with respect to star ship A and the masses m and m_o are equal. A similar story can be told for star ship B. This means that we can identify the mass m_o as the *rest mass* or *proper mass* (both terms are used), which is the mass possessed by a body when it is stationary in the frame of reference where it is measured. Thus, like momentum, mass becomes the familiar quantity in the classical limit. The proper mass is obviously a constant, but the mass of an object in motion varies with its velocity, thus it is termed the *relativistic mass* and it always exceeds the proper mass. Indeed, as (u_x, u_y, u_z) approaches the speed of light, γ_u approaches infinity and relativistic mass and the momentum does also. This implies that when accelerating a body continuously, the relativistic mass tends to infinity as the speed of light approaches. Proceeding beyond this speed therefore becomes impossible.

14. Relativity and Newton's Laws of Motion

Newton's laws of motion were not obtained mathematically. They were inferred from careful observations of how physical systems actually behave. After much study, Newton defined a force as a physical action that changes a free body's state of rest or uniform motion in a straight line and the change is in the direction of the applied force (Newton's first law). He then declared the force to be equal to the rate of change of momentum (Newton's second law) and finally, that every applied force has an equal and opposite re-action, so if a force is applied to a body, the body exerts an equal force back on the source of the force (Newton's third law). These three laws are sufficient to enable a complete mathematical description of dynamical systems. They successfully did so until Einstein showed that, close to the speed of light, things would behave differently from this *classical* description.

Newton's second law is expressed mathematically as

$$\dot{\vec{p}}=\frac{d\vec{p}}{dt}=m\frac{d\vec{v}}{dt}=\vec{f}, \qquad (74)$$

in which $\vec{p}=(p_x,p_y,p_z)$ is the (classical) momentum and $\dot{\vec{p}}=(\dot{p}_x,\dot{p}_y,\dot{p}_z)$ is its rate of change, $\vec{f}=(f_x,f_y,f_z)$ is the force vector, $\vec{v}=(v_x,v_y,v_z)$ is the velocity vector and m is the mass of the body on which the force acts. Note that this equation not only quantifies the force, it also defines what it means in the frame of reference it is observed from - as something that produces the observed change. In the light of Einstein's theory we now need to devise a form of Newton's laws that is consistent with our understanding on the classical level, but which handles the effects of relativity in a natural way.

The simplest thing that can be done is replace the classical momentum in (74) by the relativistic momentum (65) in which the mass is the relativistic form given in (64). Our intention is then to establish an interpretation that is consistent across different frames of reference. Modifying (74) accordingly we have

$$\dot{\vec{p}}=\frac{d\vec{p}}{dt}=\frac{d(m_o\gamma_v\vec{v})}{dt}=\vec{f}, \qquad (75)$$

in which the term in brackets is the relativistic momentum and γ_v is given by

$$\gamma_v=1/\sqrt{1-v_x^2-v_y^2-v_z^2} \qquad (76)$$

Equation (75) is a general expression which formally can be applied to any frame of reference. However we will need to take care to interpret it correctly in each frame, as we shall see.

Returning to our star ship theme, we set up the following scenario. Star ships A and B are in the co-linear arrangement shown in Figure 3 with B cruising at constant velocity v relative to A. Star ship B then switches on the interstellar drive and at that instant experiences a force and undergoes an acceleration. How is the physics of this event observed in the frames of reference of star ship A and B ?

Let us start with star ship A. We first note that from this frame of reference star ship B has the velocity v along the x-direction with zero y and z components due to the co-linear arrangement. This is therefore the velocity at the instant the interstellar drive is activated. We now write equation (75) as

$$\vec{f}=\frac{d\left(m_o\gamma_v\vec{v}\right)}{dt}, \qquad (77)$$

in which all quantities are defined with respect to the frame A and m_o is here the proper mass of star ship B. Proceeding with the differentiation with respect to time gives

$$\vec{f}=m_o\gamma_v\dot{\vec{v}}+m_o\gamma_v^3\left(\vec{v}\cdot\dot{\vec{v}}\right)\vec{v}, \qquad (78)$$

where $\dot{\vec{v}}=\left(\dot{v}_x,\dot{v}_y,\dot{v}_z\right)$ is the observed acceleration vector of star ship B. This is a general result, but since the velocity vector of star ship B when the drive is engaged is actually $\vec{v}=\left(v,0{,}0\right)$ equation (78) then gives the force components as

$$\begin{aligned} f_x&=m_o\gamma_v^3\dot{v}_x\\ f_y&=m_o\gamma_v\dot{v}_y.\\ f_z&=m_o\gamma_v\dot{v}_z \end{aligned} \qquad (79)$$

Note that in (79) the force $\vec{f}=\left(f_x,f_y,f_z\right)$ is not necessarily in the same direction as the acceleration $\dot{\vec{v}}=\left(\dot{v}_x,\dot{v}_y,\dot{v}_z\right)$, which is contrary to Newton’s laws.

We now consider the same event observed from the frame of star ship B. We shall ignore for the moment that in this frame of reference star ship B has no initial velocity and write

$$\vec{f}\,'=\frac{d\left(m_o\gamma_{v'}\vec{v}\,'\right)}{dt}, \qquad (80)$$

which we recognise as the equation of motion of an object of mass m_o in the frame of B. Taking the derivative leads to

$$\vec{f}\,'=m_o\gamma_{v'}\dot{\vec{v}}\,'+m_o\gamma_{v'}^3\left(\vec{v}\,'\cdot\dot{\vec{v}}\,'\right)\vec{v}\,', \qquad (81)$$

Now however, we must recognise that the 'object' we are concerned with is the star ship B itself, which has no initial velocity in its own frame of reference i.e. $\vec{v}\,'=\left(0{,}0{,}0\right)$ which means that $\gamma_{v'}=1.$ This allows us to write (81) as

$$\begin{aligned} f_x'&=m_o\dot{v}_x'\\ f_y'&=m_o\dot{v}_y'.\\ f_z'&=m_o\dot{v}_z' \end{aligned} \qquad (82)$$

in which $\dot{\vec{v}}'=(\dot{v}_x', \dot{v}_y', \dot{v}_z')$ is the acceleration experienced by star ship B in its own frame of reference.

We are now in a position to determine the relationship between the force components (f_x, f_y, f_z) and (f_x', f_y', f_z') i.e. the transformation between $\vec{f}$ and $\vec{f}'$. To do this we exploit equation (60) which provides the transformation between $\dot{\vec{v}}'$ and $\dot{\vec{v}}$ and write equation (82) as

$$\begin{aligned} f_x' &= m_o \gamma_v^3 \dot{v}_x \\ f_y' &= m_o \gamma_v^2 \dot{v}_y . \\ f_z' &= m_o \gamma_v^2 \dot{v}_z \end{aligned} \tag{83}$$

Comparing this to equation (79) leads to the conclusion that

$$\begin{aligned} f_x &= f_x' \\ f_y &= \gamma_v^{-1} f_y', \\ f_z &= \gamma_v^{-1} f_z' \end{aligned} \tag{84}$$

which is the transformation of the instantaneous force observed in the accelerating frame of reference B to the force observed in the inertial frame of reference A. The transformation can be applied at *any* instant in the acceleration of star ship B, though at different instants the velocity of star ship B will be different and so the Lorentz parameter γ will also be different. What equation (84) tells us is that the instantaneous effect of a force (f_x', f_y', f_z') applied in the moving frame is equivalent to the force (f_x, f_y, f_z) in the stationary frame.

15. Mass, Motion and Energy

When a body is in motion it has a property known as *kinetic energy*. This is a form of energy that the body acquires when a force is applied to it over a fixed distance. We shall consider the case where the force is applied to body that is originally stationary in our (inertial) reference frame and along the x-axis, so that nothing happens in the y- and z- directions and we can ignore them.

Mathematically the kinetic energy is given by the integral

$$\Delta K = \int_{s_1}^{s_2} f \, ds, \tag{85}$$

where f is the force and s is the distance moved in the direction of the force. ΔK is then the change in kinetic energy of the body as it moves from position s_1 to s_2 in a straight line along the x-axis. According to Newton's second law, force is equal to the rate of change of momentum, so (85) becomes

$$\Delta K=\int_{s_1}^{s_2}\left(\frac{dp}{dt}\right)ds=\int_{u_1}^{u_2}\left(\frac{dp}{du}\right)\left(\frac{ds}{dt}\right)du=\int_{u_1}^{u_2}\left(\frac{dp}{du}\right)udu\,, \tag{86}$$

in which p is the relativistic momentum and we have used the chain rule to convert the integral over s into an integral over velocity u. The integration limits are now from velocity u_1 to velocity u_2. Using integration by parts changes the integral to

$$\Delta K=[pu]_{u_1}^{u_2}-\int_{u_1}^{u_2}p\,du\,. \tag{87}$$

To integrate the remaining integral we must first express the momentum p in terms of velocity, thus

$$\int_{u_1}^{u_2}p\,du=m_o\int_{u_1}^{u_2}\gamma_u u du=m_o\int_{u_1}^{u_2}\frac{u}{\sqrt{1-u^2/c^2}}du=-m_o c^2\left[\sqrt{1-u^2/c^2}\right]_{u_1}^{u_2}, \tag{88}$$

where we have opted to express γ_u using ordinary units instead of light units. Thus equation (87) becomes

$$\Delta K=[pu]_{u_1}^{u_2}+m_o c^2\left[\sqrt{1-u^2/c^2}\right]_{u_1}^{u_2}. \tag{89}$$

We now set the integration limits as $u_1=0$ and $u_2=v$, in which case ΔK becomes the total kinetic energy, K, arising from the force f. So (89) becomes

$$K=pv+m_o c^2\sqrt{1-v^2/c^2}-m_o c^2. \tag{90}$$

Multiplying the second term right by $\sqrt{1-v^2/c^2}$ and then dividing by the same allows cancellation of the pv term, leading to the result

$$K=\frac{m_o c^2}{\sqrt{1-v^2/c^2}}-m_o c^2=mc^2-m_o c^2=(m-m_o)c^2. \tag{91}$$

This equation shows clearly that the kinetic energy gained by the body is manifested as a change in the body mass – the difference between the relativistic mass and the rest mass. From this we learn that mass and energy (of which kinetic energy is but one form) are inter-convertible. We can rearrange (91) into the form

$$mc^2=K+m_o c^2, \tag{92}$$

from which we identify the left hand side as the *total energy*, E, of the body, which includes the kinetic energy plus the quantity $m_o c^2$. This leads to the equation

$$E=mc^2, \tag{93}$$

in which the mass m is the *relativistic* mass. Note that, when the body is stationary i.e. $v=0$, then $K=0$ and we can write

$$E = E_o = m_o c^2. \tag{94}$$

In the latter case, E_o is the *rest energy* or the energy intrinsic to all matter. Equation (93) is arguably the most famous equation in all physics.

Note that if we had chosen to work in light units, the equation (95) would have been

$$E = m, \tag{95}$$

which reveals that in light units mass and energy are apparently the same thing.

When equation (91) is applied in the classical limit (i.e. as $v \to 0$) then it can be approximated by

$$K = m_o c^2 \left(1 + v^2/2c^2\right) - m_o c^2 \quad \text{or} \quad K = m_o v^2/2, \tag{96}$$

which is the well known classical formula for kinetic energy.

There are other relationships between energy, mass and momentum that are of use in relativity theory. A well known example is

$$E^2 = m_o^2 c^4 + p^2 c^2 \quad \text{or} \quad E^2 = E_o^2 + p^2 c^2, \tag{97}$$

which is derived as follows, starting from equation (93) we have:

$$\begin{aligned}
E^2 &= m^2 c^4 = \frac{m_o^2 c^4}{\left(1 - v^2/c^2\right)} \\
E^2 &= \frac{m_o^2 c^4}{\left(1 - v^2/c^2\right)} - p^2 c^2 + p^2 c^2 \\
E^2 &= \frac{m_o^2 c^4 - m_o^2 v^2 c^2}{\left(1 - v^2/c^2\right)} + p^2 c^2 \\
E^2 &= \frac{m_o^2 c^4 \left(1 - v^2/c^2\right)}{\left(1 - v^2/c^2\right)} + p^2 c^2 \\
E^2 &= m_o^2 c^4 + p^2 c^2
\end{aligned} \quad . \tag{98}$$

Equation (97) finds application in particle physics, where fundamental particles are accelerated up close to light speed. It represents the energy available to transform the particles into new forms. It makes one wonder what would happen if two star ships collided!

16. Relativity and Space Travel

Since we have constructed relativity theory from the perspective of star ships, it is of interest to ask what the theory means for interstellar flight. This is an important subject, particularly if you believe interstellar travel is the destiny of the human race!

Let us begin with some basic issues.

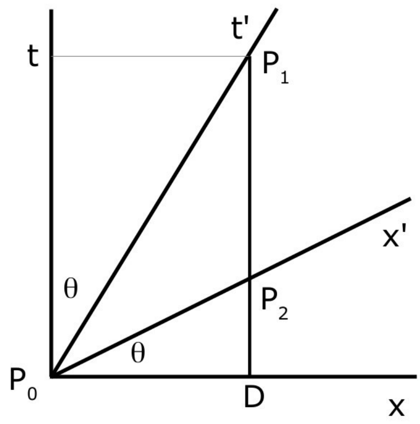


Figure 9: Travel in Space-time

a) Basic Star Ship Cruising

In Figure 9 we show the frames of reference of our two star ships A and B. Axes t, x and t', x' to refer to frames A and B respectively, in the usual co-linear configuration. We imagine star ship A is stationary with respect to a star at a distance D in the frame A. In what follows we will, for convenience, define the reference frame of star ship A as the *stationary* frame and the reference frame of star ship B as the *moving* frame. We will also use terms such as *stationary distance* or *moving time* to reflect the frame in which these are measured.

In Figure 9 we identify three space-time events: $P_{0,}P_1$ and P_2, which have the following significance:

1. P_0 is the event where star ship B coincides with A as it passes by and defines the mutual coordinate origin for frames A and B i.e. $P_0=(0,0)$ in both frames.

2. P_1 Is the event where star ship B reaches the destination star, which is where the world line of ship B (line P_0 – P_1) intercepts that of the star which is the line D – P_1. In the frame A, we record this as the coordinate $P_1=(t,D)$, where t is the time B takes to travel the distance D (according to frame A). Note that $v=D/t$ by definition.

3. P_2 marks the event where the world-line of the star crosses the x' axis. The significance of this event is that the distance P_0 - P_2 is the distance star ship B records as the distance to the star at time t'=0. As time progresses the world-line of the star moves closer to the world-line of star ship B, indicating that the distance between the two is getting shorter. The coordinates of P_2 in the frame A are $(Dt\,an\theta,D)$ which, since $tan\theta=v$, are (Dv,D).

We now transform these events to the frame of B using the Lorentz transformation

(13). Evidently $P_0'=(0,0)$ by construction. Transforming $P_1=(t,D)$ gives:

$$\begin{aligned} t' &= \gamma(t-vD)=\gamma t\left(1-v^2\right)=t\gamma^{-1} \\ D' &= \gamma(D-vt)=\gamma(D-D)=0 \end{aligned} \quad . \tag{99}$$

The second of these equations merely confirms that at the time t in frame A (or time t' in frame B), star ship B has reached the destination star. The first equation shows that time dilation occurs since $t'<t$. The travel time experienced by the ship B is thus less than than the travel time according to the reference frame A. Indeed, close to the speed of light, the journey time may be negligible!

Transforming event P_2 gives

$$\begin{aligned} t' &= \gamma(Dv-vD)=0 \\ D' &= \gamma\left(D-v^2D\right)=\gamma D\left(1-v^2\right)=D\gamma^{-1} \end{aligned} \quad . \tag{100}$$

The first of these equation shows that event P_2 lies on the x' axis i.e. $(t'=0)$, as expected. The second shows that $D'<D$. So, as far as star ship B is concerned, the distance to the destination star is less than the distance measured in the stationary frame i.e. it is Lorentz-Fitzgerald contracted, as expected.

Now, taking t' from equation (99) and D' from (100) we can write:

$$\frac{D'}{t'}=\frac{D\gamma^{-1}}{t\gamma^{-1}}=\frac{D}{t}=v, \tag{101}$$

Thus the reduced distance measured in the frame B and the time dilation compensate for each other and return the true speed with which B travels towards the destination star.

The time dilation experienced by star ship B is sometimes used as an argument in favour of interstellar space travel. Though it remains fundamentally true that space flight faster than the speed of light is not possible, time dilation means that any journey can, in principle, be accomplished in the lifetime of an astronaut. However, in the stationary frame across which the astronaut travels, extremely long time scales are inevitable, even for near-light-speed travel. Astronauts undertaking such a journey would not be able to return in a reasonable time. The world they left behind could well be lost in a distant past and there may be little sense in coming back!

b) Warp Speed!

A useful concept for relativistic travel is the idea of *warp speed* – something that conveys a sense of faster-than-light travel, but is nevertheless true to believable physics. This is something familiar to followers of the 'Star Trek' television series that boldly imagined a future where interstellar travel was routine. It seems however that warp speed does not have a proper definition, which is something we can fix here. Taking what we know about relativity theory we propose that the warp speed, ω, be defined as[12]

$$\omega=\gamma v=\frac{v}{\sqrt{1-v^2/c^2}}, \quad \text{or} \quad \omega=\frac{v}{\sqrt{1-v^2}}, \text{ (in light units)} \tag{102}$$

where γ is the Lorentz factor given in equation (12) and v is the real speed of the star ship (relative to the stationary frame of the Solar System[13]) as it travels to a destination star. The inverse of (102) gives the real speed as a function of ω i.e.

$$v=\frac{\omega}{\sqrt{1+\omega^2/c^2}}, \quad \text{or} \quad v=\frac{\omega}{\sqrt{1+\omega^2}}, \quad \text{(in light units)} \tag{103}$$

An interesting aspect of (103) is that the warp speed can be many times larger than the speed of light, but (103) can never return a real speed v that exceeds light speed.

Why is (102) a good definition of warp speed? One answer is that when v is a constant for the journey, it is equal to the distance of travel, D, (measured in the stationary frame), divided by the travel time T' (measured in the moving frame) i.e.

$$\omega=D/T', \tag{104}$$

which is the *effective* speed of travel. This is more helpful than the real speed v when planning the journey. For example, suppose you wanted to make your journey in a specified time T'. Using (104) gives the warp speed ω required to do that. The real speed v can then be obtained from (103) (which is arguably something only the ship's navigator needs to know!). What if you later want to halve the journey time? Well, just double the warp speed ω and that will do it. Note that simply doubling v would not work - it might even give a speed that exceeds the speed of light, which would be physical nonsense.

Along with the warp speed it is also useful to define a *warp distance,* d, which is expressed as

$$d=\omega t', \tag{105}$$

where t' is the time elapsed since the journey began (as measured in the moving frame of the star ship). The warp distance is then the distance travelled in the stationary frame. In the limit of $t'=T'$ (the journey time in the moving frame) then we have the expected result

$$D=\omega T', \tag{106}$$

12 Note that in light units ω is very nearly equal to γ for v close to light speed (i.e. 1).

13 For convenience we will always assume the Solar System represents our stationary reference frame for any interstellar journey and our destination star is also stationary in this frame. In our galaxy the relative velocity of any star is likely to be minute in comparison with light speed.

which is the journey distance as measured in the stationary frame. This simple relationship resembles the one we are most familiar with in ordinary circumstances: the distance travelled is the product of the speed and the time.

Of course this equation doesn't tell us anything more than the similar form $d'=vt'$, which uses the real velocity and the distance measured in the moving frame, but it has nicer properties. Mathematically d and ω represent a *linearisation* of the effects of relativity and can be used with consistency. For example doubling the warp speed ω doubles the warp distance travelled in a given time t'. This is the case even though a change in ω means the rate at which time elapses on board the star ship also changes. Thus the variables d and ω are much more convenient to work with than d' and v, even though the latter are perfectly valid variables. Once the value of v has been obtained by observation, the corresponding value of γ yields the warp speed ω, via (102) and the warp distance d can then be calculated from (105). This is all the information necessary to describe the progression of the journey. Travellers on board may simply regard their journey distance as D and their speed as ω, since it accords with 'common sense', and let the on board computer handle all the relativistic issues!

Finally, note that the following equations are analogous to (106) and are correct for journeys in which the warp speed changes with time:

$$D=\int_0^{T'} \omega\, dt', \quad \text{or} \quad D=\int_0^{T} \frac{\omega}{\sqrt{1+\omega^2}}dt. \tag{107}$$

c) Artificial Gravity?

There is no gravity on board a star ship cruising through space. Indeed, if the interstellar drive is disengaged, the ship is technically in 'free fall' and except for tidal effects due to passing objects, gravity is nullified. For long journeys, this is harmful to the human body. A related issue is the acceleration of a star ship toward light speed – how long should this take? A 'jump' to warp speed in a short time is potentially dangerous, as the force of acceleration could be lethal to all on board. However a proper application of relativity theory can provide a solution to both these problems.

In developing his General Theory of Relativity Einstein proclaimed his famous *Principle of Equivalence*, which states that an accelerating frame of reference is equivalent to the force of gravity. In other words gravity and acceleration are the same phenomenon. It follows that if the star ship was confined to a rate of acceleration, g, which responds to Earth's gravitational field, an astronaut would experience normal gravity, at least for the duration of the acceleration. We might therefore adopt this approach: apply the acceleration g for the first half of an interstellar journey and for the second half apply an equivalent deceleration $-g$. Then, with the exception of the short interval in between, normal gravity would apply throughout the voyage. Let us explore this idea.

Aboard the star ship the interstellar drive applies a force $\vec{f}'=(m_o g, 0, 0)$ in the direction of motion, with m_o being the rest mass of the ship. In the stationary frame this force transforms as $\vec{f}=\left(m_o\gamma_v^3\dot{v},0,0\right)$, following equation (83), where $\dot{v}$ is the ship's acceleration, $\gamma_v=\left(1-v^2\right)^{-1/2}$ is the Lorentz factor and v is the instantaneous velocity. Since $f_x'=f_x$ by equation (84), we can write the equation of motion for the star ship in the stationary frame as

$$\dot{v}=\frac{dv}{dt}=\gamma_v^{-3}g, \quad \text{i.e.} \quad \frac{dv}{\left(1-v^2\right)^{3/2}}=gdt. \quad \text{(in light units)} \tag{108}$$

Integrating (108) we obtain

$$\frac{v}{\sqrt{\left(1-v^2\right)}}=+c_o, \tag{109}$$

where c_o is a constant we can define if we know v at time $t=0$. We shall simply set $c_o=0$, because the star ship is initially stationary. (Note that the left hand side of (109) is the warp speed ω, as was defined in equation (102), which here increases linearly with stationary time, just as the Newtonian velocity would.) Rearranging (109) gives us

$$v=\frac{gt}{\sqrt{1+g^2t^2}}. \tag{110}$$

This equation describes how the velocity of the star ship develops as it accelerates in the stationary frame. Note that $v<1$ always (in light units), meaning it cannot exceed the speed of light.

If the destination star is at a distance D from the Earth, we can define the half-way distance as

$$\frac{D}{2}=\int_0^{t_{1/2}} v\,dt=\int_0^{t_{1/2}} \frac{gt}{\sqrt{\left(1+g^2t^2\right)}}dt \tag{111}$$

where $t_{1/2}$ is the time when the ship is at the half-way mark and we have used the velocity formula (110). Integrating (111) gives

$$\frac{D}{2}=\frac{\sqrt{\left(1+g^2t_{1/2}^2\right)}-1}{g}, \tag{112}$$

which on rearranging gives

$$t_{1/2}=\frac{D}{2}\sqrt{1+4/Dg}. \tag{113}$$

This is the time taken to get half way to the destination star. This is the point at which the ship must start to decelerate by reversing the stellar drive so that $\dot{v}'=-g$. The on board gravity 'flips' at this point so the ship is best turned around for deceleration if the astronauts wish to avoid walking on the ceiling!

Evaluating (113) for a trip to μ-Cephei, (Herschel's Garnet star), which is estimated to be 6000 light years away[14], and setting Earth's gravity $g=1.032648$ l_y/y^2 (which is equivalent to 9.81 m/s^2) gives a value for $t_{1/2}$ that is just ~ 12 days short of $3{,}001$ years! Doubling this gives the journey time as almost 6002 years. Over the whole journey, the star ship's average speed is 99.97% of light speed. The maximum speed reached is at the time $t_{1/2}$ which from equation (110) turns out to be $0.99999995\times c$. This corresponds to a warp speed of $3099\times c$. Incidentally, we report in passing that the constant application of the acceleration g achieves 98% of light speed in about 5 years of stationary time, or less than 0.1% of the μ-Cephei journey time.

The journey time aboard the star ship would be much less than 6002 years, which can be calculated as follows. From the derivative form of the Lorentz transformation (52) we have

$$dt'=dt\gamma\left(1-vu_x\right)=dt\gamma\left(1-v^2\right)=dt\sqrt{1-v^2}, \tag{114}$$

where we have used the fact that u_x and v are the same in this case, since both represent the forward velocity of the star ship. Into (114) we insert the velocity formula (110) and after some rearrangement obtain the integral

$$t_{1/2}'=\int_0^{t_{1/2}'} dt'=\int_0^{t_{1/2}} \frac{dt}{\sqrt{1+g^2t^2}}, \tag{115}$$

where $t_{1/2}'$ is the on board time required to get to the half way point. If we use the substitution $tan\theta=gt$ we can reduce (115) to

$$t_{1/2}'=\frac{1}{g}\int_{\theta_0}^{\theta_1} sec\,\theta d\theta, \tag{116}$$

in which $\theta_0=0$ and $\theta_1=\tan^{-1}gt_{1/2}$ are the integration limits. This is a standard integral with the solution

$$t_{1/2}'=\frac{1}{2g}\left[\log\left(\frac{1+sin\theta}{1-sin\theta}\right)\right]_{\theta_0}^{\theta_1}. \tag{117}$$

Since $tan\theta=,$ then $sin\theta=gt/\sqrt{1+g^2t^2}$, so (117) becomes, after rearrangement,

$$t_{1/2}'=\frac{1}{2g}\log\left(\frac{1+gt_{1/2}/\left(1+g^2t_{1/2}^2\right)^{1/2}}{1-gt_{1/2}/\left(1+g^2t_{1/2}^2\right)^{1/2}}\right)=\frac{1}{2g}\log\left(\frac{1+v_{1/2}}{1-v_{1/2}}\right), \tag{118}$$

in which $v_{1/2}$ is the real velocity of the star ship at time $t_{1/2}$ given by (110).

14 Who would not want to get a good view of this famous, dazzling, red star?

Inserting into this formula our previous values for g and $t_{1/2}$ gives the value for the half-time, $t_{1/2}'$, measured aboard the star ship as 8.46 years, which means a total journey time of 16.91 years. This is somewhat longer than the one year journey discussed previously, but this is the price of having on-board gravity! In the context of a journey to μ-Cephei, 6000 light years away, it doesn't look unreasonable.

Similar calculations to those above can be done for other stellar distances. The following table gives an idea of what's possible, based on the equations above.

Distance D (l_y)	Journey Time T′ (y)	Avg. Warp Speed ω/c
5	3.77	1.33
10	4.85	2.06
50	7.71	6.48
100	9.02	11.09
250	10.77	23.21
500	12.11	41.29
1000	13.44	74.40
2500	15.22	164.3
5000	16.56	301.9
10000	17.90	558.6
100000	22.36	4472.3
1000000	26.82	37285.6

The results are frankly astonishing - a journey of 1 million light years distance only takes ~ 7 times longer than one of 5 light years! This is not surprising given that the journey achieves an average warp speed of $37286 \times c$, while the maximum real speed differs from the speed of light by less than 1 part in 10^{11}. If such distances are accessible in less that one third of a human lifetime, it seems quite plausible that humanity could have an interstellar destiny. However there are other issues, which we consider below, which will detract from that optimistic assessment.

d) The Energy Problem

Interstellar flight implies travel at a velocity approaching that of light. What kind of energy resources does that imply? A basic calculation might go something like this.

Firstly, we determine the distance D from the Earth to the destination star. Then we decide what is an acceptable journey time $T',$ which is the time the journey will take aboard the star ship. Typically this might be the order of one to a few years. This allows us to calculate the warp velocity ω using equation (104). The real velocity v required by the star ship is then obtained using equation (103), from which the Lorentz factor γ may be obtained by simply using

$$\gamma=\omega/v, \tag{119}$$

which is obtained by reordering equation (104).

Now, according to equation (91) the kinetic energy of a star ship of mass m_o travelling with a velocity v is

$$K=m_o c^2(\gamma-1). \tag{120}$$

This is the energy (in normal units) that that the star ship possesses at the required velocity v. This energy must therefore be supplied by the interstellar drive and since no engine has ever been devised that is 100% efficient, equation (120) can only define the minimum amount of energy required, a fact we should remember in what follows.

Suppose the destination star is again μ-Cephei 6000 light years away and that we wish to go there in 1 year. From (104) we obtain a warp velocity of 6000 times the speed of light and from (104), we find the required star ship velocity v is just 4.16 m/s *less* than the speed of light. Calculating γ using equation (119) it then follows from (120) that the amount of energy required is equivalent to ~ 5999 times the rest mass, $m_0,$ of the star ship! The energy requirement of a star ship thus represents a huge technological challenge, to say the least. No known technology is capable of delivering anything like this.

However, it is possible that a star ship could somehow pick up the required energy as it travels, for instance by gathering hydrogen from the interstellar medium and converting this into energy by a presently unknown nuclear process[15], since no other physical process we know of to date is capable of generating energy on the required scale. We explore some aspects of this in the next section.

e) The Interstellar Medium

15 Given the energy demands of space flight it seems likely that the annihilation of nuclear matter is required , rather than mere nuclear fusion.

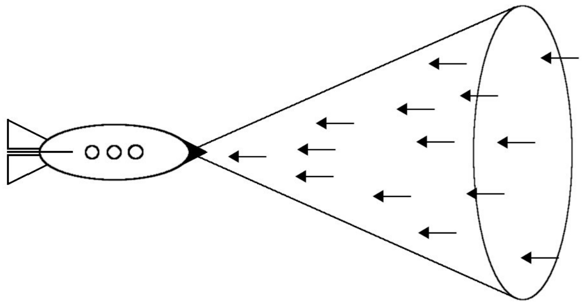


Figure 10: Sweeping the Interstellar Medium

The interstellar medium reputedly contains something of the order of 10^6 atoms of hydrogen per cubic metre. This is indeed a potential source of energy for interstellar travel, but it turns out to be both a help and a hindrance, as we will show here. (Note we shall be using normal units in preference to light units throughout what follows.)

We consider a trip on a star ship employing artificial gravity and examine the possibility of generating the necessary energy to power the journey by exploiting the availability of interstellar hydrogen. To this end we suppose our star ship has some form of 'sweeper' that gathers the hydrogen in flight to use subsequently as nuclear fuel. Crudely we imagine the sweeper to be a forward-facing, cylindrical 'funnel' (Figure 10) constructed perhaps from an electro-magnetic field, down which hydrogen atoms flow into the interstellar drive, where mass-energy is converted into thrust, supposedly like some sort of hyper jet engine (we are speculating wildly here!) As noted above, we cannot assume that the interstellar drive is capable of converting *all* the mass-energy of the hydrogen it can gather directly into the forward kinetic energy of the star ship. Some inefficiencies will inevitably arise. Furthermore, a drive that is perhaps based on rocket or jet engine principles must necessarily project some mass backwards from the star ship to generate forward momentum, so at least some of the gathered hydrogen must be used to this purpose.

In the stationary frame of the Earth we ask how much hydrogen would the sweeper gather as the star ship travels through space with an acceleration sufficient to generate on board gravity. Clearly, if it follows a straight path, the star ship sweeps up a cylindrical volume of space on its journey, so we can easily estimate the mass of hydrogen atoms it would gather as it travels a distance δl measured in the stationary frame. We express it as the equation

$$\delta M = \frac{1}{4}\pi\rho\, B^2\,\delta l, \qquad (121)$$

where δM is the mass of the hydrogen gathered over the distance $\delta l, B$ is the diameter of the sweeper and ρ is the density of the ambient hydrogen, which we assume is a constant of interstellar space. The right hand side of this equation contains some constant terms which we gather into a parameter χ:

$$\chi = \frac{1}{4}\pi\rho B^2. \tag{122}$$

From (121) and (122) the amount of mass gathered per unit of time δt is therefore

$$\frac{\delta M}{\delta t} = \chi \frac{\delta l}{\delta t}, \quad \text{or} \quad \dot{M} = \chi v, \quad \text{as} \quad \delta t \to 0. \tag{123}$$

where v is the star ship's velocity. To allow for the fact that not all the hydrogen mass gathered will be converted purely into the star ship's kinetic energy, we introduce the dimensionless parameter, $0 < \alpha \leq 1$, which specifies the fraction that is converted, and then rewrite (123) as

$$\dot{M} = \alpha \chi v. \tag{124}$$

This equation expresses the rate at which mass is converted into the kinetic energy of the star ship. The rate of energy generation (i.e. the power of the thrust) is therefore

$$\dot{E} = \alpha \chi c^2 v. \tag{125}$$

This is formally equivalent to the time derivative of the star ship kinetic energy K defined in equation (120), which is

$$\dot{K} = \frac{dK}{dt} = m_0 \gamma^3 v \dot{v}. \tag{126}$$

Since $\dot{E}$ and $\dot{K}$ are supposed equal, we combine equations (125) and (126) and cancel the common term v to give

$$m_0 \gamma^3 \dot{v} = \alpha \chi c^2. \tag{127}$$

From equation (79) we can see that the left hand side of (127) is the rate of change of momentum and the right side is therefore the force propelling the star ship, as determined in the stationary frame, which by equation (84) is the same as the force acting in the moving frame. The term $\alpha \chi c^2$ is therefore the constant force of propulsion accelerating the star ship. The instantaneous acceleration experienced on board the star ship is therefore obtained from equation (82) as

$$\dot{v}' = \alpha \chi c^2 / m_0. \tag{128}$$

Now, the on board gravity requires this acceleration should equal g, the acceleration of Earth's gravity. Combining equations (122) and (128) we have after rearrangement

$$B = \frac{2}{c}\sqrt{\frac{m_0 g}{\alpha \pi \rho}}, \tag{129}$$

which tells us what the diameter of the sweeper needs to be to gather the required amount of hydrogen. We now work this out for some reasonable values of the variables and parameters appearing.

The acceleration constant has the value $g=9.81$ m/s^2, the interstellar density of hydrogen we take as $\rho=1.66054019\times10^{-21}$ kg/m^3 and we assume a star ship rest mass of $m_0=10^6$ kg or 1000 metric tons. The parameter α we arbitrarily set to 0.25 for an order-of-magnitude test. Putting these values into (129) yields a value $B=578.6$ km. This is a large diameter for the sweeper, but it is not utterly ridiculous if the sweeper is electromagnetic in form. There are however, more troubling things to consider.

The first objection that can be raised is that the actual density of hydrogen in space can vary greatly from the quoted figure and this will obviously impact on what is possible. Some compensation for this lies in changing the parameter B, which amounts to an adjustment of the electromagnetic field. Secondly, the parameter α may in practice turn out to be very much less than $0.25,$ which potentially could have enormous impact. Once again some adjustment of parameter B would help. Thirdly, we are ignoring the fact that the sweeping up of interstellar hydrogen has serious additional effects. The atoms of the gas have a relativistic momentum capable of destroying the nuclear matter from which the ship is composed - and this does not even consider the possibility of collisions by dust particles, micro-meteorites and the like. Some means of mitigating all these is absolutely essential and the possibility of adapting the sweeper technology for the purpose seems appropriate. The last issue we must discuss is the effect of any "drag force" that may arise from the hydrogen gas as the star ship pushes its way through the interstellar medium. Despite the fact that the hydrogen density in space is extremely low, if there there is sufficient of it to power the interstellar drive it cannot be dismissed on that account.

A star ship with a sweeper encounters hydrogen gas at a mass rate, $\dot{M}$, given by equation (123). Though a fraction α of this mass is converted into the kinetic energy of the ship and is the source of the star ship's drive, the total mass gathered by the sweeper must also give rise to a drag force, since (we assume) it is initially stationary in space but is caught up by the sweeper and so acquires the speed of the star ship. The change in its momentum means the star ship must apply a force, and the reaction force experienced by the ship is the drag. We shall now calculate this.

For a small amount of gas of mass δM the change its momentum δp in a time interval δt (measured in the stationary frame of the interstellar gas) is

$$\delta p=\delta M\gamma v, \qquad (130)$$

which equations (121) and (122) permit us to write as

$$\delta p=\chi\gamma v\delta l. \qquad (131)$$

The rate of change of momentum in the time interval δt is therefore

$$\frac{\delta p}{\delta t}=\chi\gamma v\frac{\delta l}{\delta t}, \quad \text{or} \quad \dot{p}=\chi\gamma v^2, \quad \text{as} \quad \delta t\to 0. \qquad (132)$$

This is the force the star ship exerts on the interstellar gas and the drag force has the same magnitude but opposite sign. We can now incorporate the drag force into the equation of motion (127) and write

$$m_0\gamma^3\dot{v}=\chi\left(\alpha c^2-\beta\gamma v^2\right), \tag{133}$$

where we have introduced a new parameter $0\leq\beta\leq 1,$ in which β is a 'streamlining' factor that assumes clever engineering can mitigate to some extent the effect of the drag force. Equation (133) is our new equation of motion for the star ship in the stationary frame.

Before attempting to solve this equation, we first recognise that the inclusion of the drag term implies a limiting speed, v_{max}, for the star ship, since increasing v on the right hand side from zero eventually reduces the acceleration $\dot{v}$ to zero. Setting $\dot{v}=0$ in (133) leads to the equation

$$\beta\gamma v^2=\alpha c^2. \tag{134}$$

Squaring both sides of this and expanding γ gives

$$\beta^2 v^4+\alpha^2 v^2 c^2-\alpha^2 c^4=0, \tag{135}$$

which may be solved for v^2. Clearly, if $\beta=0,$ the result is $v=c$, which means there is no practical upper limit to the warp speed. However, if $\beta>0,$ the result of (135) is

$$v_{max}=c\,\frac{\alpha}{\beta}\sqrt{\frac{\sqrt{1+4\beta^2/\alpha^2}-1}{2}}. \tag{136}$$

The ratio α/β is therefore a key factor in assuring v_{max} is as large as possible. For example, if β is very small in comparison with $\alpha,$ it can easily be shown that $v_{max}\sim c$ as we would hope. But if we set $\beta=1,$ (implying there is no streamlining,) then in the case where $\alpha=1,$ (an ideal case implying full conversion of hydrogen to kinetic energy i.e. maximum power,) this equation reduces to

$$v_{max}=c\sqrt{\frac{\sqrt{5}-1}{2}}\approx 0.786\times c. \tag{137}$$

This is a rather pessimistic result - $0.786\times c$ is a warp speed of $1.272\times c$. This is far from the warp speed required to colonise a galaxy!

We may solve equation (135) in a different way, which is to obtain an estimate for the value of β compatible with interstellar travel. To this end we seek to find a value that makes possible maximum warp speeds of (say) $\omega_{max}=1000\times c,$ which, from (103) means that $v_{max}=0.9999995\times c.$ Rearranging (135) gives the equation

$$\frac{\beta}{\alpha}=\frac{c^2\sqrt{\left(1-v_{max}^2/c^2\right)}}{v_{max}^2}, \tag{138}$$

from which, with the above value of v_{max}, we find $\beta/\alpha=10^{-3},$ which gives some indication of how much clever engineering may be required to fulfil the concept of interstellar flight! What these results show is that any form of drive that requires the capture of hydrogen fuel must inevitably introduce a drag on the forward motion and any drag force will inevitably place a limit on the maximum speed attainable by the star ship.

The solution to the equation of motion (133) may be expressed in the integral form

$$y(v(t))=\int_{v(0)}^{v(t)}\left(\frac{\gamma^3}{1-\beta\gamma v^2/\alpha c^2}\right)dv=\left(\frac{\alpha\chi c^2}{m_0}\right)t. \tag{139}$$

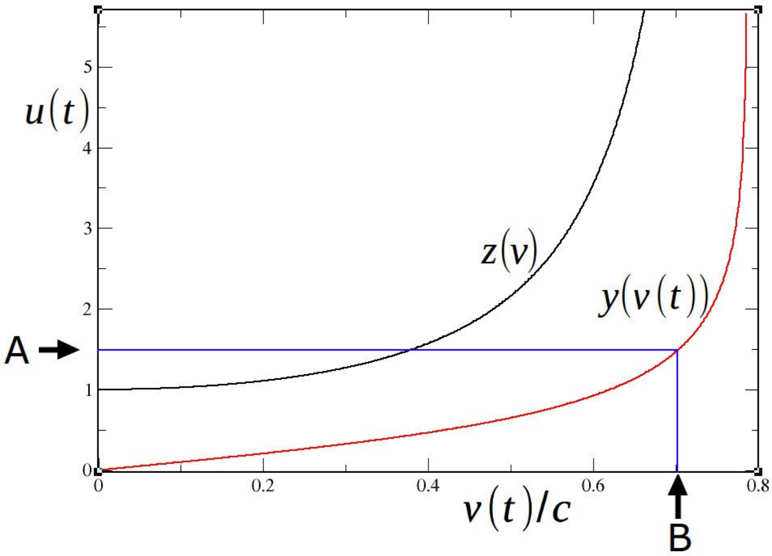
Where $y(v(t))$ is defined as a new function of the velocity $v(t)$ at time t, and $v(0)$ is the initial velocity of the star ship (which we may assume to be zero). This equation is valid for as long as $v(t)\leq v_{max}$, since equation (135) has shown that the velocity thereafter is fixed at v_{max}, otherwise integration of (139) describes the change in v as a function of time.

Clearly $v(t)$ is not a linear function, so our solution here is graphical. The function $z(v)=\gamma^3/(1-\beta\gamma v^2/\alpha c^2)$ is plotted *versus* v and integrated numerically over the range $v=0$ to $v=v_{max}$ to obtain the curve $y(v(t))$ which we plot *versus* $v(t)/c$. Next the function $u(t)=(\alpha\chi c^2/m_0)t$ is calculated for the time t. This quantity should, according to (139), equal $y(v(t))$ at the time t. So locating the point with value u along the y curve enables the determination of $v(t)$ by interpolation. The procedure is laid out in Figure 11, which shows the functions $z(v)$, $y(v(t))$ and the variable $u(t)$, as described above and assuming $\alpha=\beta=1$. The figure shows also the interpolation to find the velocity $v(t)$. A given time t is represented by $u(t)$ at point A and the corresponding velocity by the value of $v(t)/c$, which is read at B. Once $v(t)$ has been obtained for many times, the distance travelled may be calculated as the area under the v versus t curve i.e.

$$D(t)=\int_0^t v(t)\,dt. \tag{140}$$

Hopefully, this is not all academic. Though it is true a star ship may experience a severe drag force that limits the maximum velocity attainable, some means may be found to extend the application of these equations far beyond the limit set by $\beta=1$.

Figure 11



Finally we should note that artificial gravity vanishes once the acceleration drops to zero, which is another reason to ensure β is negligible. However, for as long as the star ship is travelling at a velocity less than v_{max}, it is still accelerating, and in principle this provides some on-board gravity. In fact we can always ensure that the on-board gravity holds at the value g by following the same argument that led from equation (127) to (129), but starting from the new equation of motion (133). This leads to the following result for the diameter B of the star ship sweeper

$$B=\sqrt{\frac{4m_0 g}{\pi\rho\left(\alpha c^2-\beta\gamma v^2\right)}}. \tag{141}$$

This result shows that the sweeper diameter is now dependant on the velocity v. As v increases the diameter must also increase - in a way that becomes more demanding the closer v gets to v_{max}. Presumably there will be a limit on the size of sweeper the star ship can produce - if only in terms of its power requirement (which we have not considered so far), but within that unknown limit, the gravity requirement will be met subject to the uniformity and omnipresence of the hydrogen density ρ. Against these uncertainties we can only continue to stress the need for high efficiency in energy generation $(\alpha\sim 1)$ and streamlining $(\beta\sim 0)$. The best hope is the emergence of new physics and new engineering that takes us beyond these presently understood limitations. There the matter must rest.

f) Summary

The theory presented here suggests that travel between stars in the span of one human life may be possible, though in many cases the journey can be one way only. The technological requirements are extremely challenging. The energy generation required is on an unprecedented scale, implying complete mass annihilation as the energy source. Interstellar hydrogen is a potential fuel source, but its availability is variable and collision with interstellar atomic matter threatens the structural integrity of the space ship. In addition the drag force arising from the interstellar matter represents a serious limitation on the speed attainable and could render the whole idea impractical if it cannot be overcome.

Appendix. Going Beyond the Co-linear Arrangement

In reality, star ships will approach each other in space in all sorts of ways and the co-linear arrangement that we have used in this essay is very unlikely. They will usually pass by travelling in completely different directions and their paths are very unlikely to actually cross and indeed could be any distance apart at the point of their closest approach. Nevertheless, the physics of the encounter will be very much as we have described it in the co-linear arrangement. All the predicted effects will be there, but with the added complication of the different geometry of the encounter, which will add its own effects. These are non-relativistic in nature and arise purely from a change of perspective. In this section we will describe the general geometry of star ship encounters and how this can be related to the co-linear arrangement. In the co-linear arrangement the relativistic effects can be determined and then cast onto a form relevant to each star ship.

We start with two star ships, A and B, moving at a constant velocity $\vec{v}$ relative to each other in 3D space. We will first construct an interim frame of reference from the perspective of star ship A which we shall call the t*ransfer* frame. This transfer frame will act as a mathematical bridge between the star ships. The situation is shown in Figure A1.

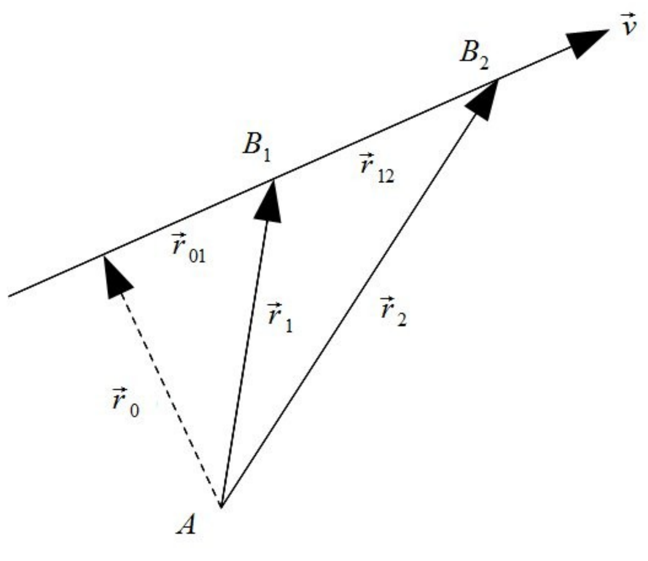


Figure A1: Construction of the Transfer Frame

Figure 9 shows star ship A at some position in space, which A naturally assumes to be at rest. From this position star ship B is seen to travel along the velocity vector $\vec{v}$ such that at time t_1 it is observed by A to be at position B_1 and at time t_2 it is observed to be at position $B_{2.}$ Star ship A records these positions as the vectors $\vec{r}_1$ and $\vec{r}_2$ respectively, which are defined with respect to the local frame of reference of A. Vectors $\vec{r}_1$ and $\vec{r}_2$ define a flat plane in 3D space and in fact everything we need to know can be calculated in this plane.

The first task is to determine the point of closest approach of B to A. This point is defined by the vector $\vec{r}_0$ which is perpendicular to the velocity vector $\vec{v}$ and starts from A and ends just touching the path followed by B. We also define two more vectors: $\vec{r}_{01}$ and $\vec{r}_{12}$. Vector $\vec{r}_{01}$ represents the travel of B from the point of its closest approach to A to where it was first located at point B_1. Vector $\vec{r}_{12}$ represents the travel from point B_1 to point B_2. Since the vectors $\vec{r}_1$ and $\vec{r}_2$ are known, then $\vec{r}_{12}$ is given by

$$\vec{r}_{12}=\vec{r}_2-\vec{r}_1. \tag{142}$$

Using this, the vector $\vec{r}_{01}$ can be determined from

$$\vec{r}_{01}=\left(\frac{\vec{r}_1\cdot\vec{r}_{12}}{r_{12}}\right)\frac{\vec{r}_{12}}{r_{12}}, \tag{143}$$

where, on the right, the term in brackets represents the length of the vector $\vec{r}_{01}$ and the remaining term represents its direction. The variable r_{12} represents the length of vector $\vec{r}_{12}$. The product written as $\vec{r}_1\cdot\vec{r}_{12}$ is known as the scalar product of $\vec{r}_1$ and $\vec{r}_{12}$. It can be written as

$$\vec{r}_1\cdot\vec{r}_{12}=r_1^x r_{12}^x+r_1^y r_{12}^y+r_1^z r_{12}^z. \tag{144}$$

It follows that vector $\vec{r}_0$ can be obtained from

$$\vec{r}_0=\vec{r}_1-\vec{r}_{01}, \tag{145}$$

or

$$\vec{r}_0=\vec{r}_1-\left(\frac{\vec{r}_1\cdot\vec{r}_{12}}{r_{12}}\right)\frac{\vec{r}_{12}}{r_{12}}. \tag{146}$$

Note that all the vectors $\vec{r}_0,\vec{r}_{01},\vec{r}_{12}$ and $\vec{v}$ are all in the plane defined by $\vec{r}_1$ and $\vec{r}_2$.

We are now in a position to define the transfer frame for star ship A. First we define the origin of this frame, which is the point from which all distance measurements will be taken, to be the point $\vec{r}_0$. We then define three vectors of unit length $\vec{e}_1,\vec{e}_2,\vec{e}_3$, as follows

$$\begin{aligned}\vec{e}_1&=\vec{r}_{01}/r_{01}\\ \vec{e}_2&=\vec{r}_0/r_0\\ \vec{e}_3&=\vec{e}_1\times\vec{e}_2\end{aligned}\ , \tag{147}$$

where r_0 and r_{01} are the lengths of vectors $\vec{r}_0$ and $\vec{r}_{01}$ respectively. The third of these equations is the so-called *vector product* of $\vec{e}_1$ and $\vec{e}_2$. It can be written as

$$\vec{e}_1 \times \vec{e}_2 = \left(e_1^y e_2^z - e_1^z e_2^y\right)\vec{i}_1 + \left(e_1^z e_2^x - e_1^x e_2^z\right)\vec{i}_2 + \left(e_1^x e_2^y - e_1^y e_2^x\right)\vec{i}_3 \qquad (148)$$

where $\vec{i}_1, \vec{i}_2, \vec{i}_3$ are the three orthogonal unit vectors that define the reference frame of A. (For this reason they are said to be the *basis* vectors of the reference frame of A.) The following points are evident from these definitions. Firstly, $\vec{e}_1$ points in the same direction as $\vec{r}_{01}$ and $\vec{r}_{12}$, which is also the direction of $\vec{v}$. Secondly, $\vec{e}_2$ is perpendicular (or *orthogonal*) to $\vec{e}_1$, since it has the same direction as $\vec{r}_0$, which was constructed to be perpendicular to $\vec{v}$. Thirdly, both $\vec{e}_1$ and $\vec{e}_2$ lie in the same plane as $\vec{r}_1$ and $\vec{r}_2$. Finally $\vec{e}_3$ is orthogonal to both $\vec{e}_1$ and $\vec{e}_2$ by construction. Thus the three unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, define an orthogonal frame of reference. It is this frame we call the transfer frame.

We now imagine a general position vector $\vec{r}$ (i.e. one pointing in any direction in 3D space) defined in the reference frame of star ship A and ask how this can be represented in the transfer frame. We start with how the vector is described in the frame of A. It is generally written as

$$\vec{r} = x\vec{i}_1 + y\vec{i}_2 + z\vec{i}_3. \qquad (149)$$

where $\vec{i}_1, \vec{i}_2, \vec{i}_3$ are the three orthogonal unit vectors that define the reference frame of A. The three variables x, y, z are the components of the vector $\vec{r}$. Note that, because the vectors $\vec{i}_1, \vec{i}_2, \vec{i}_3$ are of unit length and orthogonal to each other, we can calculate the distance of the position $\vec{r}$ from star ship A using Pythagoras' theorem i.e.

$$r^2 = x^2 + y^2 + z^2, \quad \text{or} \quad r = \sqrt{x^2 + y^2 + z^2}. \qquad (150)$$

In order to represent the vector $\vec{r}$ in the transfer frame, the first thing we note is that the origin of the reference frame of A is different from the origin of the transfer frame. The origin of the A reference frame is located at the position $\vec{O} = (0,0,0)$, while the origin of the transfer frame is at $\vec{r}_0$. So, if the star ship A records the position of an object as the vector $\vec{r}$, the position vector of the same object recorded from the point $\vec{r}_0$ will be different. If we call this vector $\vec{s}$, it is related to $\vec{r}$ through the expression

$$\vec{s} = \vec{r} - \vec{r}_0. \qquad (151)$$

The vector $\vec{s}$ locates the object with respect to the origin of the transfer frame, but note that it is still described in the reference frame of star ship A. This is made clear by writing the above equation in an expanded form:

$$\vec{s} = (x - x_0)\vec{i}_1 + (y - y_0)\vec{i}_2 + (z - z_0)\vec{i}_3 \qquad (152)$$

where the components of $\vec{s}$ (appearing in the brackets) are explicitly written in terms of the components of $\vec{r}$ and $\vec{r}_0$.

What we need to do now is re-write the vector $\vec{s}$ in a form compatible with the transfer frame. This means we must describe $\vec{s}$, not in terms of $\vec{i}_1,\vec{i}_2,\vec{i}_3$, but in terms of $\vec{e}_1,\vec{e}_2,\vec{e}_3$, for which the components will be different. To do this we must find out how to describe the vectors $\vec{i}_1,\vec{i}_2,\vec{i}_3$ in terms of the vectors $\vec{e}_1,\vec{e}_2,\vec{e}_3$. It turns out this is quite easy to do. Just as any vector can be described with respect to any set of basis vectors, $\vec{i}_1,\vec{i}_2,\vec{i}_3$ can be described with respect to $\vec{e}_1,\vec{e}_2,\vec{e}_3$. This is represented by the following

$$\begin{aligned}\vec{i}_1&=R_{11}\vec{e}_1+R_{12}\vec{e}_2+R_{13}\vec{e}_3\\ \vec{i}_2&=R_{21}\vec{e}_1+R_{22}\vec{e}_2+R_{23}\vec{e}_3.\\ \vec{i}_3&=R_{31}\vec{e}_1+R_{32}\vec{e}_2+R_{33}\vec{e}_3\end{aligned} \tag{153}$$

These three equations describe how each basis vector $\vec{i}_1,\vec{i}_2,\vec{i}_3$ is composed of contributions from the basis vectors $\vec{e}_1,\vec{e}_2,\vec{e}_3$. Each coefficient R_{jk} that appears represents the *component* a vector $\vec{i}_j$ with respect to a basis vector $\vec{e}_k$. These coefficients are calculated as follows

If we take the first equation and form the scalar product of both sides with the vector $\vec{e}_1$ we have

$$\begin{aligned}\vec{i}_1\cdot\vec{e}_1&=(R_{11}\vec{e}_1+R_{12}\vec{e}_2+R_{13}\vec{e}_3)\cdot\vec{e}_1,\\ \vec{i}_1\cdot\vec{e}_1&=R_{11}\vec{e}_1\cdot\vec{e}_1+R_{12}\vec{e}_2\cdot\vec{e}_1+R_{13}\vec{e}_3\cdot\vec{e}_1,\\ \vec{i}_1\cdot\vec{e}_1&=R_{11}.1+R_{12}.0+R_{13}.0,\end{aligned} \tag{154}$$

where we have used the properties: $\vec{e}_1\cdot\vec{e}_1=1,\vec{e}_1\cdot\vec{e}_2=0,\vec{e}_1\cdot\vec{e}_3$, or in general $\vec{e}_j\cdot\vec{e}_j=1$, and $\vec{e}_j\cdot\vec{e}_k=0, if(j\neq k)$. These are the consequence of the vectors $\vec{e}_1,\vec{e}_2,\vec{e}_3$ being of unit length and orthogonal to each other. Finally we have

$$R_{11}=\vec{i}_1\cdot\vec{e}_1. \tag{155}$$

We can now take the first equation again and form the scalar product with vector $\vec{e}_2$. In the same way as before we get the result

$$R_{12}=\vec{i}_1\cdot\vec{e}_2. \tag{156}$$

By repeating this procedure, forming the scalar product of all the equations (153) with all of the vectors $\vec{e}_1,\vec{e}_2,\vec{e}_3$, we can determine all the coefficients R_{jk}. The general result is

$$R_{jk}=\vec{i}_j\cdot\vec{e}_k. \tag{157}$$

This defines all the coefficients.

Now that we have a way of describing vectors $\vec{i}_1,\vec{i}_2,\vec{i}_3$ in terms of vectors $\vec{e}_1,\vec{e}_2,\vec{e}_3$, we are able to express the vector $\vec{s}$ in terms of $\vec{e}_1,\vec{e}_2,\vec{e}_3$ also. We replace each vector $\vec{i}_j$ with appropriate expression in $\vec{e}_1,\vec{e}_2,\vec{e}_3$ and obtain

$$\vec{s} = (x - x_0)(R_{11}\vec{e}_1 + R_{12}\vec{e}_2 + R_{13}\vec{e}_3) \\ +(y - y_0)(R_{21}\vec{e}_1 + R_{22}\vec{e}_2 + R_{23}\vec{e}_3) \\ +(z - z_0)(R_{31}\vec{e}_1 + R_{32}\vec{e}_2 + R_{33}\vec{e}_3) \quad . \tag{158}$$

We now gather together terms particular to each vector $\vec{e}_k$ to obtain

$$\vec{s} = ((x - x_0)R_{11} + (y - y_0)R_{21} + (z - z_0)R_{31})\vec{e}_1 \\ +((x - x_0)R_{12} + (y - y_0)R_{22} + (z - z_0)R_{32})\vec{e}_2 \\ +((x - x_0)R_{13} + (y - y_0)R_{23} + (z - z_0)R_{33})\vec{e}_3. \tag{159}$$

The terms in the curled brackets are of course the components of the vector $\vec{s}$ in the transfer frame, which can be seen if we write $\vec{s}$ as

$$\vec{s} = s_x\vec{e}_1 + s_y\vec{e}_2 + s_z\vec{e}_3, \tag{160}$$

with

$$s_x = (x - x_0)R_{11} + (y - y_0)R_{21} + (z - z_0)R_{31}, \\ s_y = (x - x_0)R_{12} + (y - y_0)R_{22} + (z - z_0)R_{32}, \\ s_z = (x - x_0)R_{13} + (y - y_0)R_{23} + (z - z_0)R_{33}. \tag{161}$$

This set of equations represents the transformation of the coordinates in the frame of A to the coordinates in the transfer frame. It should be noted that the same transformation can be applied to any vector like $\vec{s}$, which is drawn from the point at $\vec{r}_0$ to another point in 3D space. All that changes in the above formulae are values of x, y and z.

We now have two important results. Firstly, we know how to calculate the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, which define the transfer frame of star ship A and secondly, we know how to convert any set of coordinates determined in the frame of A into the coordinates in the transfer frame. In other words we can convert any vector in frame A to the corresponding vector in the transfer frame.

There is one other piece of information we need: the origin of time. It is most convenient to define the zero time as being when star ship B is at the point of closest approach to A. This is exactly at the position $\vec{r}_0$. Since we know it is at position $\vec{r}_1$ at time t_1 and we know its velocity is $\vec{v}$, we can calculate the time it was at $\vec{r}_0$ from

$$t_0 = t_1 - r_{01}/v. \tag{162}$$

The ratio r_{01}/v is of course the time B takes to travel along the vector $\vec{r}_{01}$. The time t_0, which is determined in the frame of A, corresponds to the time of zero in the transfer frame. So if t is the time measured in the A frame and ξ is the time measured in the Transfer frame, the two are related through

$$\xi = t - t_0. \tag{163}$$

This completes the specification of the transfer frame as constructed from the frame of star ship A. The same procedures can be employed to define the transfer frame for star ship B: Star ship B considers itself to be at rest while star ship A moves with a constant velocity we shall call $\vec{v}\,'$. The path of A can be determined from two sightings A_1 and A_2 at positions $\vec{r}\,'_1$ and $\vec{r}\,'_2$ at times t'_1 and t'_2 and it will have a point of closest approach at $\vec{r}\,'_0$ which can be determined from $\vec{r}\,'_1$ and $\vec{r}\,'_2$ in the same way as before. We can then construct the vectors $\vec{e}_1{}',\vec{e}_2{}',\vec{e}_3{}'$ and thus arrive at the transfer frame constructed from the frame of B.

It is important to note that the two transfer frames have been constructed so that each star ship can consider itself to be in a fixed position at the origin of their local coordinates and the transfer frame for each is centred on a fixed position in their local frame (indicated by $\vec{r}_0$ or $\vec{r}\,'_0$ accordingly). In each local frame a transformation is required to change coordinates from the local frame to the transfer frame, but this will be a *different* transformation for each. Neither star ship needs to know about the local transformation of the other, it only needs to know how to interpret the information from the transfer frame the other constructs. Like the original local frames the two transfer frames move at constant velocity with respect to each other, so they are equivalent and equally good for describing events in space, but we should expect them to measure space and time differently. We need the Lorentz transformation to handle the differences.

Before doing this however, let us make clear certain properties of the two transfer frames, which follow from their construction. To start with, we can immediately say how some important vectors in each frame are simply related to their counterparts, from the way the two transfer frames have been constructed. Firstly we have

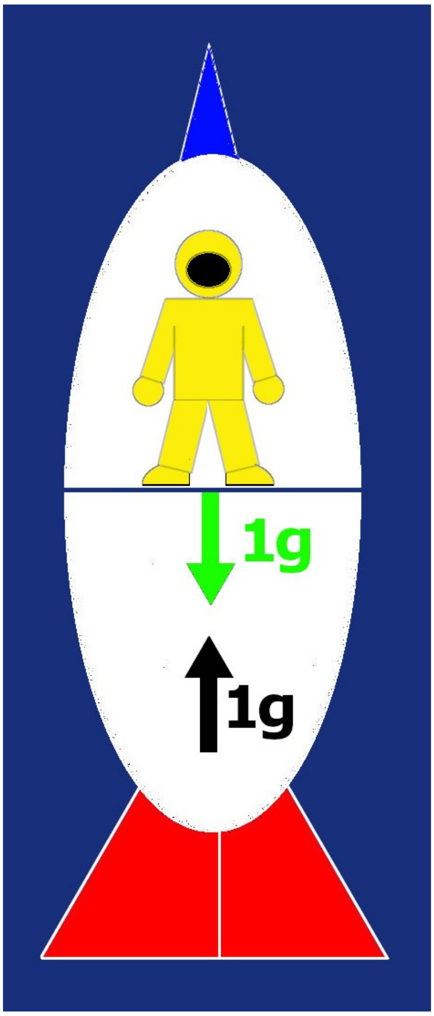
$$\vec{v}\,' = -\vec{v}, \text{ and } \vec{r}\,'_0 = -\vec{r}_0. \tag{164}$$

These follow from the way the frame is constructed in each case. The lengths of the corresponding vectors are the same in both frames. In relativity theory, each star ship must measure the scalar speed of the other to be the same (though in opposite directions). The theory also says there is no relativistic change in length scales perpendicular to the velocity vector. Another observation we can make is that the unit vectors $\vec{e}\,'_1,\vec{e}\,'_2,\vec{e}\,'_3$ must point in the opposite direction to their counterparts $\vec{e}_1,\vec{e}_2,\vec{e}_3$ i.e.

$$\begin{aligned} \vec{e}\,'_1 &= -\vec{e}_1, \\ \vec{e}\,'_2 &= -\vec{e}_2, \\ \vec{e}\,'_3 &= -\vec{e}_3. \end{aligned} \tag{165}$$

This must be so because the vector $\vec{e}_1$ is constructed to have the same direction as vector $\vec{v}$ and likewise the vector $\vec{e}\,'_1$ has the same direction as $\vec{v}\,'$. Similarly $\vec{e}_2$ and $\vec{r}_0$ have the same direction, as do $\vec{e}\,'_2$ and $\vec{r}\,'_0$. Finally the $\vec{e}_3$ and $\vec{e}\,'_3$ vectors, being constructed from the vector products of $\vec{e}_1$ with $\vec{e}_2$ and $\vec{e}\,'_1$ with $\vec{e}\,'_2$ respectively, inevitably generate $\vec{e}_3$ and $\vec{e}\,'_3$ as opposing vectors.

We now have all the information we need to understand the relativistic properties associated with the relative velocity of the two star ships. In summary, all observations made by star ship A are transformed into corresponding observations in its transfer frame, as are the observations of star ship B into its own transfer frame. From these transfer frames the two sets of observations can be interrelated using the Lorentz transformation (13). It should be clear however, that apart from the Lorentz transformation that is applied in the transfer frame, all this work is independent of relativity theory. This justifies confining our discussion of the theory to the co-linear arrangement of Figure 3.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